Inverse Problems Exercise 11

Sommersemester 2022

Return your written solutions by email to vesa.kaarnioja@fu-berlin.de by Monday 18 July, 2022, 12:15

Please note that there will be no in-person lecture or exercise session on Monday 18 July. The final lecture will be uploaded as a recording on the course website on Monday 18 July.

1. The true state of a time-varying system is $v_k^* = 4 + \sin(0.03k)$, $k = 1, \ldots, 500$. The *observation model* is

$$y_k = v_k + \eta_k, \quad \eta_k \sim \mathcal{N}(0, 0.1^2).$$

Using the random walk *evolution model*

$$v_{k+1} = v_k + \xi_k, \quad \xi_k \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, \gamma^2)$$

consider the following tasks:

   (a) Simulate the measurements using v_k^* and implement the Kalman filter algorithm to compute the estimates $\mathbb{E}[v_k | y_1, \ldots, y_k]$, $k = 1, \ldots, 500$. Use the value $\gamma = 0.1$ and an initial distribution $v_0 \sim \mathcal{N}(m_0, \sigma_0^2)$, where $m_0 = 4$ and $\sigma_0 = 1$.

   (b) Run the Kalman filter with different combinations of m_0, σ_0, and γ and make inferences about their effects on the behavior of the estimated state.

2. Let us revisit the heat equation example

$$\begin{cases} \partial_t u(x,t) = \partial_x^2 u(x,t) & x \in (0,1), t \in (0,T), \\ \partial_x u(0,t) = \partial_x u(1,t) = 0 & t \in (0,T), \\ u(x,0) = f(x) & x \in (0,1) \end{cases}$$

considered during the lecture.

Suppose that the initial heat distribution $f\colon [0,1] \to \mathbb{R}$ is uncertain and the task is to track the solution $u(x, t_j)$ over the discrete time instances $t_j = j\Delta t$, $j \in \{1, \ldots, J\}$ and $\Delta t = T/J$, given noisy measurements at the interval endpoints

$$y_j = \begin{bmatrix} u(0,t_j) \\ u(1,t_j) \end{bmatrix} + \eta_j, \quad \eta_j \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, \gamma^2 I_2), \quad j \in \{1, \ldots, J\},$$

where $\gamma > 0$ and $I_2 \in \mathbb{R}^{2\times 2}$ is an identity matrix.

The file `kfdemo.m` on the course page `http://numerik.mi.fu-berlin.de/wiki/SS_2022/InverseProblems.php` implements a simple Kalman filter to solve the corresponding evolution-observation model

$$\begin{aligned} \boldsymbol{u}(t_{j+1}) &= M\boldsymbol{u}(t_j) + \xi_j, \quad \xi_j \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, \sigma^2 I),\ j = 0, 1, \ldots, J-1, \\ y_{j+1} &= H\boldsymbol{u}(t_{j+1}) + \eta_{j+1}, \quad j = 0, 1, \ldots, J-1, \end{aligned}$$

by imposing a smoothness prior on the uncertain initial heat distribution f and approximating the evolution model with the backward Euler method. Here, $\boldsymbol{u}(t) := [u(x_1, t), \ldots, u(x_{n-1}, t)]^{\mathrm{T}}$ for $x_j := j/n$, $j = 1, \ldots, n-1$.

Implement an *ensemble Kalman filter* to solve this problem by modifying the program `kfdemo.m`. Test your algorithm by using different ensemble sizes, e.g., $N = 10^k$ for $k = 2, 3, 4, \ldots$. How does increasing the ensemble size affect the reconstructed solution?