Inverse Problems | Exercise 2
Sommersemester 2022
Return your written solutions either in person or by email
to vesa.kaarnioja@fu-berlin.de by Monday 9 May, 2022, 12:15

1. Let $A \in \mathbb{R}^{m\times n}$, let $I \in \mathbb{R}^{n\times n}$ be the identity matrix, $\delta > 0$, and

$$K = \begin{bmatrix} A \\ \sqrt{\delta} I \end{bmatrix} \in \mathbb{R}^{(m+n)\times n}.$$

Show that the singular values of K satisfy

$$\lambda_j \geq \sqrt{\delta}, \quad j = 1, \ldots, n.$$

2. Let $A \in \mathbb{R}^{m\times n}$, $y \in \mathbb{R}^m$, and let $x_\delta \in \mathbb{R}^n$ be the Tikhonov regularized solution of $Ax = y$, i.e., x_δ minimizes the functional

$$\|Ax - y\|^2 + \delta\|x\|^2, \quad \delta > 0.$$

Let us define the "discrepancy function" $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ by

$$f(\delta) = \|Ax_\delta - y\|^2.$$

Prove that

$$f'(\delta) = 2\delta\langle x_\delta, (A^{\mathrm{T}}A + \delta I)^{-1}x_\delta\rangle = 2\delta x_\delta^{\mathrm{T}}(A^{\mathrm{T}}A + \delta I)^{-1}x_\delta.$$

In particular, note that f is monotonically increasing. How is this result useful from the viewpoint of the Morozov discrepancy principle?