Inverse Problems | Exercise 3

Sommersemester 2022

Return your written solutions either in person or by email to vesa.kaarnioja@fu-berlin.de by Monday 16 May, 2022, 12:15

1. Let $A \in \mathbb{R}^{m\times n}$ and $y \in \mathbb{R}^m$, and consider the equation $Ax = y$. Prove that the corresponding Landweber–Fridman iterates $\{x_k\}_{k=0}^{\infty}$ can be written explicitly as

$$x_k = \beta \sum_{j=0}^{k-1} (I - \beta A^{\mathrm{T}} A)^j A^{\mathrm{T}} y, \quad k = 1, 2, \ldots .$$

Moreover, with the help of a singular system (λ_j, v_j, u_j) of A, show that this is equal to

$$x_k = \sum_{j=1}^{p} \frac{1}{\lambda_j} (1 - (1 - \beta\lambda_j^2)^k)(u_j^{\mathrm{T}} y) v_j, \quad k = 0, 1, \ldots,$$

where $p = \mathrm{rank}(A)$.

2. Let us consider a simple X-ray tomography problem with limited angle data. On the course's homepage at

`http://numerik.mi.fu-berlin.de/wiki/SS_2022/InverseProblems.php`

you can download the file `week3.mat`, which can be imported into MATLAB with the command

```
load week3 A S N
```

The file contains a sparse tomography matrix $A \in \mathbb{R}^{4900\times 10000}$, a noisy sinogram $S \in \mathbb{R}^{70\times 70}$, and the dimension of the original object $N = 100$.

Left-multiplying a (vectorized) square image $B \in \mathbb{R}^{10000}$ by matrix A computes 70 discretized X-ray projections of B from a limited angle view of $\pm 60°$ with respect to the positive horizontal axis. For example, the code

```
B = phantom(100); % Shepp-Logan phantom with size 100x100
figure;
imagesc(B), axis square, colormap gray
y = A*B(:);
Z = reshape(y,70,70); % The corresponding sinogram
figure;
imagesc(Z), axis square, colormap gray
```

produces the sinogram corresponding to object `B`. The first column of the resulting `Z` is the X-ray projection of the phantom from the angle of view of $-60°$ with respect to the positive horizontal axis, the last column of `Z` corresponds to the angle of view of $+60°$ with respect to the positive horizontal axis, and the remaining columns of `Z` are the X-ray projections of the phantom from equidistant angles of view between these two extremes.

Your task is to reconstruct the object corresponding to the given sinogram S. First, form the vectorized sinogram by `y = S(:)`, then use Landweber–Fridman iteration to solve the equation

$$Ax = y.$$

Use $\beta = 3$ and the Morozov discrepancy principle with

$$\varepsilon = \sqrt{0.01^2 \cdot 70^2} = 0.70,$$

which is the square root of the expected value for the squared norm of the (vectorized) noise, under the assumption that each pixel of the sinogram S is contaminated with normally distributed additive noise with zero mean and standard deviation 0.01. Visualize the resulting reconstruction after using `X = reshape(x,100,100)` to reshape the reconstruction into an image, and plot the value of the residual

$$f(k) := \|Ax_k - y\|$$

as a function of k. Here, $\{x_k\}$ denote the Landweber–Fridman iterates. Why is $\beta = 10$ a bad choice? How about $\beta = 0.01$?