Inverse Problems Exercise 4
Sommersemester 2022
Return your written solutions either in person or by email
to vesa.kaarnioja@fu-berlin.de by Monday 23 May, 2022, 12:15

1. Let $B \in \mathbb{R}^{n\times n}$ be a symmetric and positive definite matrix, and assume that $x \in \mathbb{R}^n$ is the solution of $Bx = w$ for some given $w \in \mathbb{R}^n$. If one approximates x using the conjugate gradient method with the initial guess $x_0 \in \mathbb{R}^n$, it is known that the k^{th} iterate satisfies (you are not required to prove this)

$$\|x - x_k\|_B \leq \left(\frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}\right)^k \|x - x_0\|_B, \quad k = 1, 2, \ldots, \tag{1}$$

where $\|z\|_B^2 = z^{\mathrm{T}}Bz$ and $\kappa = \mu_{\max}/\mu_{\min}$ is the condition number of B, i.e., it is the ratio of the largest eigenvalue $\mu_{\max}$ and the smallest eigenvalue $\mu_{\min}$ of B.

   (a) Show that $\mu_{\min}^{1/2}\|z\| \leq \|z\|_B \leq \mu_{\max}^{1/2}\|z\|$ for all $z \in \mathbb{R}^n$, where $\|\cdot\|$ denotes the standard Euclidean norm in $\mathbb{R}^n$.

   (b) Using the result in part (a), derive an error estimate in the standard Euclidean norm induced by (1). That is, derive an estimate for $\|x - x_k\|$ in terms of $\|x - x_0\|$, the condition number κ, and the iteration index k.

   (c) Let $A \in \mathbb{R}^{m\times n}$, $y \in \mathbb{R}^m$, and let $x_\delta \in \mathbb{R}^n$ be a Tikhonov regularized solution to $Ax = y$. Consider solving the corresponding normal equation

   $$(A^{\mathrm{T}}A + \delta I)x = A^{\mathrm{T}}y$$

   with the conjugate gradient method starting from some initial guess $x_0 \in \mathbb{R}^n$. Suppose that $\operatorname{rank}(A) < n$, which is a sound assumption (at least up to the numerical precision) if A corresponds to an inverse/ill-posed problem. Use part (b) to write an estimate for $\|x_\delta - x_k\|$ with the help of the largest singular value of A, i.e., $\lambda_1 = \|A\|$, the regularization parameter $\delta > 0$, the iteration index k, and the initial error $\|x_\delta - x_0\|$.

2. Let us revisit the X-ray tomography problem from last week's exercises. On the course's homepage at

   `http://numerik.mi.fu-berlin.de/wiki/SS_2022/InverseProblems.php`

   you can download the file `week3.mat`, which can be imported into MATLAB with the command

```
load week3 A S N
```

   The file contains a sparse tomography matrix $A \in \mathbb{R}^{4900\times 10000}$, a noisy sinogram $S \in \mathbb{R}^{70\times 70}$, and the dimension of the original object $N = 100$.

   Your task is to reconstruct the object corresponding to the given sinogram S, this time using the conjugate gradient method. First, form the vectorized sinogram by `y = S(:)`. Then use the conjugate gradient method with the initial guess $x_0 = 0$ to solve the normal equation

$$A^{\mathrm{T}}Ax = A^{\mathrm{T}}y.$$

Use the Morozov discrepancy principle with

$$\varepsilon = \sqrt{0.01^2 \cdot 70^2} = 0.70$$

as the stopping rule: terminate the iteration when the norm of the residual corresponding to the *original equation* is less than ε, i.e., when

$$\|Ax_k - y\| \leq \varepsilon.$$

Visualize the resulting reconstruction after using `X = reshape(x,100,100)` to reshape the reconstruction into an image, and plot the value of the residual

$$f(k) := \|Ax_k - y\|$$

as a function of k. Here, $\{x_k\}$ denote the conjugate gradient iterates. How many iterations does it take to satisfy the Morozov criterion? Visualize also the reconstruction that results from 1000 rounds of conjugate gradient iterations.