Inverse Problems Exercise 5
Sommersemester 2022
Return your written solutions either in person or by email
to vesa.kaarnioja@fu-berlin.de by Monday 30 May, 2022, 12:15

1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, where Ω is a sample space, $\mathcal{F}$ is a σ-algebra of measurable events (subsets) of Ω, and $\mathbb{P}$ is a probability measure on $\mathcal{F}$. Let $E = \bigcup_{j\in\mathcal{I}} E_j$ be a union of measurable sets $E_j \in \mathcal{F}$, $j \in \mathcal{I}$, such that $\mathbb{P}(E_j) = 0$.

   (a) Show that $\mathbb{P}(E) = 0$ if $\mathcal{I}$ is a countable index set.

   (b) Show an example where $\mathbb{P}(E) > 0$ if $\mathcal{I}$ is an uncountable index set.

2. Suppose that μ_j, $j = 1, 2, 3$, are σ-finite probability measures on the same measure space and $\mu_1 \ll \mu_2 \ll \mu_3$. Show that

$$\frac{d\mu_1}{d\mu_3} = \frac{d\mu_1}{d\mu_2}\frac{d\mu_2}{d\mu_3}, \quad \mu_3\text{-almost surely.}$$

3. Let $X \sim \mathcal{N}(x_0, C)$ be a Gaussian random variable with mean $x_0 \in \mathbb{R}^n$ and covariance matrix $C \in \mathbb{R}^{n\times n}$, which is symmetric and positive definite. What is $\mathbb{E}\,\|X - x_0\|_2^2$?

4. (a) Let $z_1 \sim \mathcal{N}(m_1, C_1)$ and $z_2 \sim \mathcal{N}(m_2, C_2)$ be independent Gaussian random variables for $m_1, m_2 \in \mathbb{R}^k$ and symmetric, positive definite covariance matrices $C_1, C_2 \in \mathbb{R}^{k\times k}$. Show that

$$z = a_1 z_1 + a_2 z_2 \sim \mathcal{N}(a_1 m_1 + a_2 m_2, a_1^2 C_1 + a_2^2 C_2).$$

   (b) Let $z \sim \mathcal{N}(m, C)$ be a Gaussian random variable for $m \in \mathbb{R}^k$ and symmetric, positive definite covariance matrix $C \in \mathbb{R}^{k\times k}$. Let $L \in \mathbb{R}^{d\times k}$ and $a \in \mathbb{R}^d$. Show that

$$w = Lz + a \sim \mathcal{N}(Lm + a, LCL^{\mathrm{T}}).$$