Inverse Problems — Exercise 6

Sommersemester 2022

Return your written solutions either in person or by email
to vesa.kaarnioja@fu-berlin.de by Monday 13 June, 2022, 12:15

Note that there is no lecture or exercise session on Monday 6 June (public holiday).

1. Consider the following multiplicative noise model

$$y_j = a_j x_j, \quad 1 \le j \le n,$$

where $y, x, a \in \mathbb{R}^n$, and assume that a is a log-normally distributed multiplicative noise vector with independent components, that is, $\log a_j \sim \mathcal{N}(\log a_0, \sigma^2)$. Furthermore, a is assumed to be independent of x. By taking the logarithm, the noise model becomes additive. Using this observation, derive the likelihood density $\mathbb{P}(y|x)$ for such $x \in \mathbb{R}^n$ that $x_j > 0$ for all $j = 1, \ldots, n$.

2. Suppose our inverse problem is given by

$$y = Ax + \eta,$$

where $y \in \mathbb{R}^k$ is the observation, $\eta \in \mathbb{R}^k$ is additive measurement noise, and $A \in \mathbb{R}^{k \times d}$ is the matrix modeling the measurement. Moreover, suppose that the noise distribution is given by $\eta \sim \mathcal{N}(0, I)$ and the prior distribution by $x \sim \mathcal{N}(0, C)$, where $C \in \mathbb{R}^{d \times d}$ is a symmetric and positive definite matrix.

   (a) Form the posterior density $\pi^y(x)$.

   (b) Notice that the MAP estimator is precisely the minimizer of $-\log(\pi^y(x))$. Using this observation, solve the MAP estimator explicitly.

Hint: In part (b), it is helpful to use the Cholesky decomposition $C^{-1} = L^{\mathrm{T}} L$.

3. The Laplace distribution is characterized by a location parameter $t \in \mathbb{R}$ and a scale parameter $b > 0$. It has the probability density

$$\pi_{t,b}(x) = \frac{1}{2b} \exp\left(-\frac{|x - t|}{b}\right).$$

Compute the Hellinger distance

$$d_{\mathrm{H}}(\pi_{t,b}, \pi_{t,c}),$$

where $b, c > 0$.