Inverse Problems Exercise 8
Sommersemester 2022
Return your written solutions either in person or by email
to vesa.kaarnioja@fu-berlin.de by Monday 27 June, 2022, 12:15

1. Consider the integral

$$\int_{[0,1]^4} \cos(x_1^2 + x_2^2 + x_3^2 + 2x_4^2)\,\mathrm{d}x.$$

Estimate the value of this integral by implementing a Monte Carlo sampler. How many samples do you need to achieve accuracy of the order 10^{-3}?

2. Suppose we are given the posterior distribution

$$\pi^y(x) = \frac{1}{Z} g(x,y)\pi(x),$$

where $x, y \in \mathbb{R}^2$, we have the prior density $\pi(x) = \frac{1}{2\pi}\exp(-\frac{1}{2}(x_1^2 + x_2^2))$, and

$$g(x,y) = \exp(-|x_1 - y_1^2| - |x_2^2 - y_2|).$$

Here, $Z = \int_{\mathbb{R}^2} g(x,y)\pi(x)\,\mathrm{d}x$.

Suppose we are given the observation $\bar{y} = (3,2)^\top$. Use importance sampling to estimate the posterior mean.

3. Consider a linear Bayesian inverse problem

$$Y = AX + \mathcal{E},$$

where all objects are finite dimensional. Suppose our prior distribution is Gaussian $\mathcal{N}(0, C_0)$ and the noise is distributed according to $\mathcal{E} \sim \mathcal{N}(0, \Gamma)$, with C_0, Γ symmetric and positive definite. Assume that a 'true' solution $x^\dagger$ exists and that we are able to obtain a sequence of *independent* measurements $\{y_j\}_{j=1}^N$, where y_j is sampled from the distribution of $Ax^\dagger + \mathcal{E}$.

   (a) What is the posterior distribution π^y of X, given $Y = y$?
   (b) Let $\bar{y} := \frac{1}{N}\sum_{j=1}^N y_j$. From what distribution is the average $\bar{y}$ generated?
   (c) Now, let us model the N measurements by

$$Y_j = AX + \mathcal{E}_j, \quad j = 1, \ldots, N,$$

using the same prior and noise distribution as before and assuming that $X, \mathcal{E}_1, \ldots, \mathcal{E}_N$ are independent. What is the joint probability density of $Y_1, \ldots, Y_N$, given $X = x$? What is the posterior distribution $\pi_N^{y_1,\ldots,y_N}$ of X, given $Y_1 = y_1, \ldots, Y_N = y_N$?
   (d) Explain what happens with $\pi_N^{y_1,\ldots,y_N}$ in the limit $N \to \infty$ if the problem is underdetermined.