Freie Universität Berlin
Prof. Dr. Ana Djurdjevac
Prof. Dr. Ralf Kornhuber

Numerics III
SS 2022

1. Exercise sheet, due May 4, 2022

Problem 1 (6 Points)
Let $\Omega \subset \mathbb{R}^3$ denote a bounded domain with sufficiently smooth boundary $\Gamma = \partial\Omega$ and outward normal n. Prove the following Green's formulas for v, $w \in C^2(\overline{\Omega})$ denoting

$$\frac{\partial v}{\partial n} = \nabla v \cdot n.$$

a) $\displaystyle\int_\Omega \Delta v \; dx = \int_\Gamma \frac{\partial v}{\partial n} \; d\Gamma$

b) $\displaystyle\int_\Omega \nabla v \cdot \nabla w \; dx = -\int_\Omega \Delta v w \; dx + \int_\Gamma \frac{\partial v}{\partial n} w \; d\Gamma$

c) $\displaystyle\int_\Omega (w\Delta v - v\Delta w) \; dx = \int_\Gamma \left(w\frac{\partial v}{\partial n} - v\frac{\partial w}{\partial n} \right) \; d\Gamma$

Hint: Use Gauß' integral theorem.

Problem 2 (6 Points) Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with sufficient smooth boundary $\Gamma = \partial\Omega$ and

$$H_{C,g} = \{ v \in C^1(\overline{\Omega}) \mid v|_\Gamma = g \}$$

with $g = u_0|_\Gamma$ and given $u_0 \in C^1(\overline{\Omega})$. We set $H_C = H_{C,0}$.
a) Show that the minimization problem

$$u \in H_{C,g} : \quad \mathcal{J}(u) \le \mathcal{J}(v) \quad \forall v \in H_{C,g} \quad \text{for} \quad \mathcal{J}(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 \; dx \tag{1}$$

is equivalent to the variational problem

$$w \in H_C : \quad \int_\Omega \nabla w \cdot \nabla v \; dx + \int_\Omega \nabla u_0 \cdot \nabla v \; dx = 0 \quad \forall v \in H_C.$$

and $u = u_0 + w$.
b) Show that (1) has at most one solution.