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Numerics III
SS 2022

10. Exercise sheet, due July 6, 2022

Problem 1 (2 + 2 + 4 + 2 TP)
We consider the variational problem

$$u \in H : \qquad a(u, v) = \ell(v) \qquad \forall v \in H \tag{1}$$

with a symmetric, H-elliptic bilinear form $a(\cdot, \cdot)$ and $\ell \in H'$. Let $\|\cdot\| = a(\cdot, \cdot)^{1/2}$ denote the energy norm, let $S \subset Q \subset H$ denote closed subspaces of H, and let $u_S \in S$ and $u_Q \subset Q$ be the Ritz-Galerkin approximations of u with respect to S and Q.

a) Show the following global lower a posteriori discretization error estimate

$$\|u_Q - u_S\| \leq \|u - u_S\|.$$

b) Show that the local lower a posteriori bound for the discretization error

$$|\ell(\lambda_p) - a(u_S, \lambda_p)| / \|\lambda_p\| \leq a_p(u - u_S, u - u_S)^{1/2}$$

holds in the special case $H = H_0^1(\Omega)$, $\ell(v) = \int_\Omega fv\, dx$ with $f \in L^2(\Omega)$,

$$a(v, w) = \int_\Omega \nabla v \cdot \nabla w \, dx, \quad a_p(v, w) = \int_{\text{supp}\lambda_p} \nabla v \cdot \nabla w \, dx$$

and the nodal basis elements λ_p, $p \in N_h$, of the space S_h of piecewise linear finite elements.

c) Show that the saturation assumption

$$\|u - u_Q\| \leq \beta \|u - u_S\| \qquad \text{with } \beta < 1$$

implies the global upper a posteriori discretization error estimate

$$\|u - u_S\| \leq C \|u_Q - u_S\| \qquad \text{with } C = \frac{1}{\sqrt{1 - \beta^2}}$$

d) Prove that there is no upper bound for the local discretization error
$a_p(u - u_S, u - u_S)^{1/2}$ in terms of local quantities, i.e. supported by supp λ_p.

Problem 2 (2 + 2 TP)
Consider the variational problem (1) with $\Omega = (0,1)$, $H = H_0^1(\Omega)$, $a(v,w) = \int_\Omega v'(x)w'(x)\,dx$ and its discretization with respect to the hierarchy of piecewise finite element spaces S_k,

$$S_k = \{v \in H_0^1(\Omega) \mid v|_t \in \Pi_1 \;\; \forall t \in T_k\}, \quad k = 1, 2, \ldots, K,$$

with nodal basis

$$\Lambda_k = \{\lambda_p^{(k)} \mid p \in N_k\}, \quad k = 1, 2, \ldots, K,$$

and associated partitions T_k,

$$T_k = \{t = (jh_k, (j+1)h_k) \mid j = 0, \ldots, 2^k - 1\}, \quad h_k = 2^{-k},$$

with nodes $N_k = \{jh_k \mid j = 1, \ldots, 2^k - 1\}$.

a) Show that the so-called 'hierarchical basis'

$$\Lambda^{(K)} = \bigcup_{k=1}^{K} \{\lambda_p^{(k)} \mid p \in N_k \setminus N_{k-1}\}$$

with $N_0 = \emptyset$ is indeed a basis of S_K.

b) Show that the hierarchical basis is orthogonal with respect to the scalar product $a(\cdot,\cdot)$.