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Numerics III
SS 2022

2. Exercise sheet, due May 11, 2022

Problem 1 (4 Points)
Let $\Omega \subset \mathbb{R}^3$ denote a bounded domain with sufficiently smooth boundary $\partial\Omega$ and outward normal n. Consider the Neumann boundary value problem

$$-\Delta\, u = f \quad \text{in } \Omega, \qquad \frac{\partial}{\partial n} u = g \quad \text{on } \partial\Omega \tag{1}$$

with $f \in C(\overline{\Omega}), g \in C(\partial\Omega)$.

a) Show that $\int_\Omega f\, dx = -\int_{\partial\Omega} g\, ds$ is a necessary condition for existence of a solution $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$.

b) Let $u_0 \in C^2(\Omega) \cap C^1(\overline{\Omega})$ be a solution of (1). Is this solution unique? If not, provide a representation of all solutions of (1).

Problem 2 (3 Points) Classify the following partial differential equations in $\mathbb{R}^2$
a) $-\mathrm{div}(\alpha(x)\nabla u) = 0, \qquad \alpha \in C^1(\mathbb{R}^2), \quad \alpha(x) \geq \alpha_0 > 0.$

b) $\varepsilon\Delta u - \vec{\beta}\nabla u = 0, \qquad$ (i) for $\varepsilon \neq 0, \quad$ (ii) for $\varepsilon = 0.$

c) $u_y^2 u_{xx} + u_x u_y u_{xy} + u_x^2 u_{yy} = u_x - u_y.$

Problem 3 (8 Points) Consider the Cauchy problem for the wave equation

$$\begin{aligned} u_{tt} - u_{xx} &= 0 && \text{in } \Omega = \mathbb{R} \times \mathbb{R}_+ \\ u(\cdot, 0) = u_0, \quad u_t(\cdot, 0) &= u_1 && \text{on } \partial\Omega = \mathbb{R}. \end{aligned} \tag{2}$$

a) Derive d'Alembert's solution of (2).

b) Provide the domain of dependence of $u(x_0, t_0)$ for some $x_0 \in \mathbb{R}$ and $t_0 > 0$, i.e. the subset $D \subset \mathbb{R}$ such that modifications of $u_0(x)$ or $u_1(x)$ for $x \in D$ would affect the value $u(x_0, t_0)$ while modifications of $u_0(x)$ or $u_1(x)$ for $x \notin D$ would not.

c) The Cauchy problem (2) is a mathematical model for an infinitely long vibrating string. Solve (2) for the initial data $u_0 = \sin(x)$, $u_1 = 0$ and provide a physical interpretation of the result.