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Numerics III
SS 2022

3. Exercise sheet, due May 18, 2022

Problem 1 (2 + 4 Points)
a) Let

$$\Omega = \mathbb{R} \times \mathbb{R}_+, \quad \mathbb{R}_+ = \{y \in \mathbb{R} \mid y > 0\}$$

with boundary $\partial\Omega = \{(x, 0) \in \mathbb{R}^2 \mid x \in \mathbb{R}\}$. Show that the Cauchy initial value problem

$$\Delta u = 0 \quad \text{in } \Omega, \quad u = g_0 \text{ and } \frac{\partial}{\partial n} u = g_1 \quad \text{on } \partial\Omega$$

with given g_0, $g_1 \in C(\mathbb{R})$ is ill-posed.
Hint: See Section 2.3.1 in the lecture notes.
b) Let $\Omega = (0, \pi) \times (0, \pi)$. Show that the boundary value problem to find

$$u \in \{v \in C(\overline{\Omega}) \mid v_{xx} \in C(\Omega),\ v_y \in C(\Omega)\}$$

such that

$$u_y = u_{xx}, \quad (x, y) \in \Omega,$$
$$u(x, 0) = \sin(x), \quad x \in (0, \pi),$$
$$u(x, y) = 0, \quad (x, y) \in \partial\Omega \setminus \{(x, 0) \mid x \in (0, \pi)\}$$

is ill-posed.

Problem 2 (4 + 2 Points)
a) Extend the representation formula in Theorem 3.8 of the lecture notes to $n = 1$ space dimension. To this end, use the notation

$$\int_{\partial[\alpha,\beta]} f(x) d\sigma(x) = f(\alpha) + f(\beta),$$

for an interval (α, β) and set the outer normals to $n(\alpha) = -1$ and $n(\beta) = 1$.
b) Derive Green's function of the first kind $G(\cdot, x)$ for $x \in [0, L]$, $L > 0$, and a resulting representation formula for smooth solutions $u \in C^2([0, L])$ of boundary value problems

$$-u'' = f \text{ in } [0, L], \quad u(0) = g(0), \quad u(L) = g(L)$$

with given $f \in C[0, L]$ and $g(0)$, $g(L) \in \mathbb{R}$. Draw a sketch of $G(\cdot, x)$ and interprete the results.