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Numerics III
SS 2022

4. Exercise sheet, due May 25, 2022

Problem 1 (2 + 2 + 2 TP)
Let $\Omega_h = \{(ih, jh) \mid i, j = 1, \ldots, n-1\}$ with $h = 1/n$, $\overline{\Omega}_h = \{(ih, jh) \mid i, j = 0, \ldots, n\}$, $\partial\Omega_h = \overline{\Omega_h} \setminus \Omega_h$ and $H_0 = \{U : \overline{\Omega}_h \mapsto \mathbb{R} \mid U(x) = 0 \; \forall x \in \partial\Omega_h\}$. Then H_0 is isomorphic to $\mathbb{R}^N$ with $N = (n-1)^2$ and it holds

$$H_0 \ni U \longleftrightarrow \underline{U} = (U(x_i))_{i=1}^N \in \mathbb{R}^N$$

with some numbering

$$\Omega_h = \{x_i \mid i = 1, \ldots, N\}. \qquad (1)$$

a) Find a numbering (1), such that

$$-\underline{\Delta_h U} = A_h \underline{U} \qquad \forall U \in H_0$$

holds with a symmetric matrix $A_h \in \mathbb{R}^{N \times N}$.

b) Show

$$4\|\underline{U}\|_\infty \le \|A_h \underline{U}\|_\infty \le \frac{8}{h^2}\|\underline{U}\|_\infty \qquad \forall \underline{U} \in \mathbb{R}^N.$$

c) Show that the condition number $\kappa(A_h) = \|A_h\|_2 \|A^{-1}\|_2$ grows with decreasing mesh size according to

$$\kappa(A_h) \le \frac{2}{h^2}.$$

Problem 2 (2 TP) Let $[a, b] \subseteq \mathbb{R}$ be an interval. Suppose $f \in C(\overline{I})$ and that $u \in C^2(\overline{I})$ is a solution of

$$-\Delta u = f, \qquad u|_{\{a,b\}} = 0.$$

Furthermore, let $N \in \mathbb{N}$ and $h := \frac{b-a}{N}$. For each $i \in \{0, \ldots, N\}$ define $x_i := a + ih$ and for every $i \in \{1, \ldots, N-1\}$ let

$$\lambda_i \colon \overline{I} \longrightarrow \mathbb{R}, \qquad x \longmapsto \begin{cases} \frac{1}{h}(x - x_{i-1}) & \text{for } x \in [x_{i-1}, x_i] \\ -\frac{1}{h}(x - x_{i+1}) & \text{for } x \in [x_i, x_{i+1}] \\ 0 & \text{for } x \notin [x_{i-1}, x_{i+1}]. \end{cases}$$

Prove for all $i \in \{1, \ldots, N-1\}$ that

$$-\Delta_h u(x_i) := -\frac{u(x_{i-1}) - 2u(x_i) + u(x_{i+1})}{h^2} = \frac{\int_I f(x)\lambda_i(x)\,\mathrm{d}x}{\int \lambda_i(x)\,\mathrm{d}x} =: f_h(x_i).$$

Hint : Multiply the PDE by λ_i, then integrate and use partial integration.

Problem 3 (8 PP)
Let $\Omega = (0,1)^2$, $f \in C(\Omega)$, $g \in C(\partial\Omega)$, $h = 1/n$, $\Omega_h = \{(ih, jh) \mid i, j = 0, \ldots n-1\}$, and $f_h = f|_{\Omega_h}$, $g_h = g|_{\partial\Omega_h}$.
a) Implement the Shortley-Weller method

$$-\Delta_h U(x(= f_h(x) \quad \forall x \in \Omega_h, \qquad U(x) = g_h(x) \quad \forall x \in \partial\Omega_h$$

by writing functions

$$\begin{aligned} A &= \mathrm{OperatorAssembler(N)} \\ B &= \mathrm{FunctionalAssembler(f, g, n)} \end{aligned}$$

which assemble the coefficient matrix $A \in \mathbb{R}^{N,N}$ and the right-hand side vector $F \in \mathbb{R}^N$ with $N = (n-1)^2$ for given n function handles f and g. The reurned matrix should be stored in a sparse data format.
b) Apply your program to the following data and corresponding exact solutions

$f_1(x) = -2(x_1(x_1-1) + x_2(x_2-1))$	$g_1 = 0$	$u_1(x) = x_1 x_2 (x_1 - 1)(x_2 - 1)$
$f_2(x) = -4$	$g_2(x) = \|x\|_2^2$	$u(x) = \|x\|_2^2$
$f_3(x) = 2\pi^2 \sin(\pi x_1)\sin(\pi x_2)$	$g_3 = 0$	$u_3(x) = \sin(\pi x_1)\sin(\pi x_2)$

and $n = 2^k$ with $k = 2, \ldots, 8$. Plot the graphs of the corresponding discrete solutions U_1, U_2, U_3. In addition, plot the discretization errors

$$e_k(h) = \max_{x \in \Omega_h} |u_k(x) - U_k(x)|$$

over $h_k = 2^{-k}$ in a logarithmic scaling. You can use the command `axis equal` and the function `loglog`.