Freie Universität Berlin
Prof. Dr. Ana Djurdjevac
Prof. Dr. Ralf Kornhuber

Numerics III
SS 2022

5. Exercise sheet, due Juni 1, 2022

Problem 1 (4 + 4 Extra-TP)
Let V be a normed linear space over $\mathbb{R}$ with norm $\|\cdot\|$.
a) Prove that the norm $\|\cdot\|$ is induced by the scalar product

$$(v, w) = \frac{1}{4}(\|v + w\|^2 - \|v - w\|^2)$$

on V, if and only if the norm $\|\cdot\|$ satisfies the parallelogram identity

$$\|v + w\|^2 + \|v - w\|^2 = 2(\|v\|^2 + \|w\|^2).$$

b) Let $\Omega \subset \mathbb{R}^d$ be a domain with Lipschitz boundary $\partial\Omega$ and consider the linear space

$$V = \{v \in C^1(\Omega) \cap C(\overline{\Omega}) \mid \|v\|_{1,\infty} < \infty\}$$

equipped with the norm

$$\|v\|_{1,\infty} = \max\{\|v\|_\infty, \||\nabla v|_\infty\|_\infty\}, \ \ \|v\|_\infty = \sup_{x\in\Omega} |v(x)|, \ \ |z|_\infty = \max_{i=1,\dots,d} |z_i|, \ z \in \mathbb{R}^d.$$

Show that there is no scalar product that induces $\|\cdot\|_\infty$.

Problem 2 (2 Extra-TP)
Let $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary $\partial\Omega$. We consider the inner product space

$$H_C = \{v \in C^1(\Omega) \cap C(\overline{\Omega}) \mid \|v\|_1 < \infty, \ v|_{\partial\Omega} = 0\}$$

with

$$\|v\|_1 = \left(\|v\|_0^2 + \||\nabla v|\|_0^2\right)^{1/2}, \ \ \|v\|_0 = \left(\int_\Omega v^2 \, dx\right)^{1/2}, \ \ |z| = \left(\sum_{i=1}^d z_i^2\right)^{1/2}, \ z \in \mathbb{R}^d$$

Give an example for a positive definite bilinear form on H_C which is not H_C-elliptic.