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Numerics III
SS 2022

6. Exercise sheet, due June 8, 2022

Problem 1 (2 TP)
Let H be a Hilbert space with scalar product $(\cdot,\cdot)$ and $S \subset H$ a closed subspace. Show that the mapping $P : H \mapsto S$ is well-defined by

$$Pu = \operatorname*{argmin}_{v \in S} \frac{1}{2}(v,v) - (u,v)$$

and that P is an orthogonal projection of H on S.

Problem 2 (4 TP)
Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary $\partial\Omega$. Show by a counterexample that the pre-Hilbert space

$$X = \{v \in C^1(\Omega) \cap C(\overline{\Omega}) \mid v|_{\partial\Omega} = 0,\ (v,v)_1 < \infty\}$$

with scalar product

$$(v,w)_1 = (v,w)_{L^2(\Omega)} + \sum_{k=1}^{d} (v_{x_k}, w_{x_k})_{L^2(\Omega)}$$

is not complete.

Problem 3 (2 + 4 + 2 TP)
a) Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary $\partial\Omega$. Assume $v \in L^2(\Omega)$ and that v_{x_k} is the weak derivative of v in the direction x_k, $k = 1,\dots,d$. Prove that the weak derivative v_{x_k} is uniquely determined.
b) Investigate the weak differentiability and, if it exists, compute the weak derivative of the following functions in $L^2(-1,1)$

$$f(x) = \sin(x), \quad g(x) = |x|, \quad h(x) = \sqrt{|x|}.$$

c) Show that the Heaviside function is not weakly differentiable in $L^2(-1,1)$.