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Numerics III
SS 2022

7. Exercise sheet, due June 15, 2022

Problem 1 (4 + 4 TP)
Let $\Omega = (a, b)$ with $a < b$ and $\partial\Omega = \{a, b\}$. We set

$$L^2(\partial\Omega) = \{v = (v(a), v(b)) : \partial\Omega \mapsto \mathbb{R} \mid \|v\|^2_{L^2(\partial\Omega)} = v(a)^2 + v(b)^2 < \infty\}.$$

a) Show that the a priori bound

$$\|v|_{\partial\Omega}\|_{L^2(\partial\Omega)} \leq C\|v\|_{H^1(\Omega)} \qquad \forall v \in C^\infty(\overline{\Omega})$$

holds with $C = 2\sqrt{\max\{(b-a), (b-a)^{-1}\}}$.
b) Show that there is a bounded linear mapping

$$\text{tr} : H^1(\Omega) \mapsto L^2(\partial\Omega)$$

with the property

$$\text{tr } v = v|_{\partial\Omega} = (v(a), v(b)) \qquad \forall v \in H^1(\Omega) \cap C(\overline{\Omega}).$$

Hint: Use that for any $v \in H^1(\Omega) \cap C(\overline{\Omega})$ there is a sequence $(\varphi_n)_{n\in\mathbb{N}} \subset C^\infty(\overline{\Omega})$ with $\varphi_n \to v$ in $H^1(\Omega)$ for $n \to \infty$ and $\varphi_n(x) = v(x)$, $x = a, b$ for all $n \in \mathbb{N}$.

Problem 2 (4 TP)
Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary $\partial\Omega$ and let

$$H^1_g(\Omega) = \{v \in H^1(\Omega) \mid \text{tr } v = g\}$$

with the trace operator $\text{tr} : H^1(\Omega) \to L^2(\Omega)$ that exists by Theorem 5.25 and given $g \in L^2(\partial\Omega)$. Assume that the bilinear form $a(\cdot,\cdot)$ is bounded on $H^1(\Omega)$ and $H^1_0(\Omega)$-elliptic and that the functional ℓ on $H^1(\Omega)$ is linear and bounded. Show that then the variational problem

$$u \in H^1_g(\Omega) : \quad a(u, v) = \ell(v) \quad \forall v \in H^1_0(\Omega)$$

has a unique solution, if and only g is in the range of tr, i.e. if $H^1_g(\Omega) \neq \emptyset$.