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Numerics III
SS 2022

8. Exercise sheet, due June 22, 2022

Problem 1 (2 TP)
Let $\Omega \subset \mathbb{R}^n$ denote a bounded Lipschitz domain, $\mathcal{C} = \{v \in H^1(\Omega) \mid v = \text{const.}\}$, $f \in \{v \in L^2(\Omega) \mid \int_\Omega v \, dx = 0\}$, and $u \in H = \mathcal{C}^\perp = \{v \in H^1(\Omega) \mid (v, w)_1 = 0 \; \forall w \in \mathcal{C}\}$ be the solution of the variational problem

$$u \in H: \quad (\nabla u, \nabla v) = (f, v) \quad \forall v \in H.$$

Show that u is a classical solution of

$$-\Delta u = f \text{ in } \Omega, \qquad \frac{\partial}{\partial n} u = 0 \text{ on } \partial\Omega,$$

provided that the regularity assumption $u \in \mathcal{C}^\perp \cap C^2(\overline{\Omega})$ holds.
Hint: Proceed in a similar way as in the proof of Theorem 1.4.

Problem 2 (2 + 4 TP)
Let $\Omega \subset \mathbb{R}^n$ denote a bounded Lipschitz domain.
a) Derive a weak formulation of the inhomogenous Neumann problem

$$-\Delta u = f \text{ in } \Omega, \qquad \frac{\partial}{\partial n} u = g \text{ on } \partial\Omega,$$

b) Derive sufficient conditions for existence, uniqueness and stability of a weak solution.

Problem 3 (4 TP)
Consider the inhomogeneous variational problem

$$u \in H: \quad (\nabla u, \nabla v) = (f, v) \quad \forall v \in H_0^1(\Omega)$$

with $H = \{v \in H^1(\Omega) \mid \text{tr } v = g\}$ and given $g \in V(\partial\Omega)$ with

$$V(\partial\Omega) = \{v \in L^2(\partial\Omega) \mid \exists w \in H^1(\Omega): \; v = \text{tr } w\}$$

equipped with the norm

$$\|v\|_{V(\partial\Omega)} = \inf_{\substack{w \in H^1(\Omega) \\ \text{tr} w = v}} \|w\|_{H^1(\Omega)}.$$

Show well-posedness in the sense that there is a unique solution $u \in H$ that depends continuously on the data $f \in L^2(\Omega)$ and $g \in V(\partial\Omega)$.