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Numerics III
SS 2022

9. Exercise sheet, due June 29, 2022

Problem 1 (2 TP)
Assume that the bilinear form is symmetric and $H_0^1(\Omega)$-elliptic. Show that the stiffness matrix A associated with a finite-dimensional subspace $S \subset H_0^1(\Omega)$ is symmetric and positive definite.

Problem 2 (2 + 2 + 2 Extra-TP)
Consider a domain $\Omega \subset \mathbb{R}^2$, the variational equality

$$u \in H_0^1(\Omega): \quad a(u,v) = \ell(v) \quad \forall v \in H_0^1(\Omega)$$

with $a(\cdot,\cdot) = (\nabla\cdot,\nabla\cdot)$ and $\ell = (f,\cdot)$, $f \in C^2(\overline{\Omega})$, a quasiuniform triangulation $\mathcal{T}_h$ which is regular in the sense of Definition 6.11 of the lecture notes, and the subspace of piecewise linear finite elements $S = S^{(1)} \subset H_0^1(\Omega)$.

a) Show that the norm equivalence

$$c\|v\|_{L^2(\Omega)} \leq \left(\sum_{\mathcal{N}_h} v(p)^2 h^2 \right)^{1/2} \leq C\|v\|_{L^2(\Omega)}$$

holds with positive constants c, $C \in \mathbb{R}$ independent of the mesh size h.
Hint: The eigenvalues of the mass matrix $M = (m_{pq})$ are located in an interval $[ch^2, Ch^2]$ with positive constants c, $C \in \mathbb{R}$ independent of the mesh size h.

b) Application of the local quadrature rule

$$Q(f\lambda_p) = \tfrac{1}{3}|\text{supp } \lambda_p| f(p) \tag{1}$$

as obtained by piecewise linear interpolation of $f\lambda_p$ leads to the perturbed functional

$$\ell_h(v) = \sum_{p\in\mathcal{N}} Q(f\lambda_p)v_p, \quad v = \sum_{p\in\mathcal{N}} v_p\lambda_p \in S_h.$$

Show that the error estimate

$$|\ell(v) - \ell_h(v)| \leq ch^2\|v\|_1$$

holds with a constant C in dependent of h.
Hint: The local quadrature error can be bounded according to

$$\left| \int_\Omega f(x)\lambda_p(x)\, dx - Q(f\lambda_p) \right| \le ch^4$$

with a constant c independent of h and $p \in \mathcal{N}$.

c) Show that the Ritz-Galerkin approximation with quadrature

$$u_h \in S: \quad a(u_h, v) = \ell_h(v) \quad \forall v \in S$$

satisfies the error estimate

$$\|u - u_h\|_1 \le C(h^2 + \min_{v \in S} \|u - v\|_1)$$

Problem 3 (2 Extra-TP)
Consider the variational problem

$$u \in H_0^1(\Omega): \quad (\nabla u, \nabla v) = (f, v) \quad \forall v \in H_0^1(\Omega) \tag{2}$$

with $\Omega = (0,1)^2$ and $f \in L^2(\Omega)$ and its approximation by piecewise linear finite elements with respect to an equidistant quadratical grid with mesh size $h = 1/n$. Show that the resulting stiffness matrix up to the factor h^2 coincides with the matrix provided by the Shortley-Weller finite difference method.

Problem 4 (8 PP)
a) Implement the finite element approximation of the solution of (2) by piecewise linear finite elements with respect to an equidistant quadratical grid with mesh size $h = 1/n$ by writing functions

$$A = \text{OperatorAssembler(N)}$$
$$B = \text{FunctionalAssembler(f, g, n)}$$

which assemble the coefficient matrix $A \in \mathbb{R}^{N,N}$ and the right-hand side vector $b \in \mathbb{R}^N$ with $N = (n-1)^2$ for given n function handles f and g. The returned matrix should be stored in a sparse data format. Apply the quadrature rule (1) for (approximate) evaluation of the integrals on the right-hand side.
b) Apply your program to the following data and corresponding exact solutions

$f_1(x) = -2(x_1(x_1-1) + x_2(x_2-1))$	$g_1 = 0$	$u_1(x) = x_1x_2(x_1-1)(x_2-1)$
$f_2(x) = -4$	$g_2(x) = x_1^2 + x_2^2$	$u(x) = x_1^2 + x_2^2$
$f_3(x) = 2\pi^2 \sin(\pi x_1)\sin(\pi x_2)$	$g_3 = 0$	$u_3(x) = \sin(\pi x_1)\sin(\pi x_2)$

and $n = 2^k$ with $k = 2, \ldots, 8$. Plot the graphs of the corresponding discrete solutions. In addition, plot the discrete L^2-norm of the discretization errors

$$e(h) = (\sum_{p \in \mathcal{N}} (u(p) - u_h(p))^2 h^2)^{1/2}$$

over various step sizes $h_k = 2^{-k}$ in a double logarithmic scaling. You could use the command `axis equal` and the function `loglog`.