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Exercise 12 for the lecture

Numerics II

WS 2011/12

Due: till Thursday, 2. February 12 o'clock

Problem 1 (3 TP)
Show that

$$\frac{1}{2}\sqrt{\kappa(A)}\ln\left(\frac{2}{\mathrm{TOL}}\right)$$

CG iterations are necessary to reduce the error by a factor TOL.

Problem 2 (5 PP + 3 bonus PP)
Consider the linear system

$$AU = b \tag{1}$$

with the symmetric positive definite matrix $A \in \mathrm{R}^{n\times n}$ and $b \in \mathrm{R}^n$.

a) Implement the conjugate gradient method and the preconditioned conjugate gradient method as `matlab` functions

```
function [u, uk] = cg(A, b, u0, tol, uexact),
```

and

```
function [u, uk] = pcg(A, b, u0, tol, uexact, pre).
```

`u`, `uk`, `A`, `b`, `u0`, `tol`, and `uexact` denote the last iterate, a vector containing all iterates, the system matrix, the right hand side, the initial iterate, the error tolerance, and the exact solution, respectively. `pre` denotes a `function` `y` = `pre(x)` that applies the inverse $y = B^{-1}x$ of some preconditioner B. The iteration should stop if the energy norm $\|\cdot\|_A = \langle A\cdot,\cdot\rangle^{\frac{1}{2}}$ of the error is smaller than the tolerance.

b) Test your programs with the matrix of the model problem given in the lecture and the right hand side $b = AU$ where U is the pointwise evaluation of $(x_1 - x_1^2)(x_2 - x_2^2)$ for `u0` $= 0$, `tol` $= 10^{-8}$ and various choices of n. Use one Jacobi step and one symmetric Gauß-Seidel step, respectivly as preconditioner for the pcg-method. Plot the error over the number of iteration steps. Compare the results with the simple Jacobi and Gauß-Seidel method.

c) Augment your function `pcg` from a) with an error estimator. For Jacobi and symmetric Gauß-Seidel preconditioned cg mathod plot the estimated error

$$\|d\|_B, \qquad d = B^{-1} r_k$$

over the number of iteration steps and compare the results with the exact error from b).

Problem 3 (4 bonus TP)
Derive from the cg method a method for *non-symmetric* A by application of A^T to $Ax = b$. Which Krylov spaces are spanned by this method? What can you say about the convergence properties?