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Exercise 3 for the lecture

Numerics II

WS 2011/12

Due: till Thursday, 17. November 12 o'clock

Problem 1 (2 TP)
Let $\Phi^t = \exp(t)$. Show that if Ψ^τ is consistent with Φ^t with order p, then

$$\Psi^\tau = R(z) = \exp(z) + \mathcal{O}(z^{p+1}) \qquad \text{for } z \to 0$$

with $z = \lambda\tau$.

Problem 2 (3 TP)
Let $\Psi^\tau : \mathrm{R}^n \to \mathrm{R}^n$ the discrete flux operator of the implicit trapezoidal rule with stepsize τ.

a) Consider the linear system
$$x'(t) = Ax(t).$$
Show that Ψ^τ can be written as
$$\Psi^\tau = R(\tau A),$$
where R is a rational function of the matrix τA.

b) Under what conditions on τ is Ψ^τ A-stable? Is asymptotic stability inherited from the continuous problem?

Problem 3 (3 TP)
Compute the Butcher scheme for the collocation method with the supporting points of the Simpson rule.

Problem 4 (4 TP)

a) Compute the time step restriction for the method of Runge $\Psi^\tau = x + \tau f(x + \frac{\tau}{2} f(x))$ applied to the linear system

$$x'(t) = -\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} x(t). \tag{1}$$

b) Plot the stability domain for the method of Runge and visualize the time step restriction for (1).

Problem 5 (6 PP)

a) Implement the (possibly implicit) Runge-Kutta method for the linear system

$$x'(t) = Mx(t), \qquad t \in (\texttt{I(1)}, \texttt{I(2)}] \qquad x(\texttt{I(1)}) = \texttt{x}_0$$

in `matlab` as `function [x, t, k] = RungeKuttaLinear(M, x0, I, tau, b, A)`, where `M`, `x0`, `I`, and `tau` denote the system matrix, the initial value, the time interval and the step size, respectively and the Butcher scheme is given by `b`, `A`. The returned values `x`, `t`, and `k` should contain the solution at each time step, the time steps, and the intermediate vectors k_i for all time steps, respectively.

b) Test your program with (1) and the initial value $x(0) = (1, 2)^T$ on the interval $(0, 5]$ with the step sizes $\tau = 10^{-3}, 10^{-2}, 10^{-1}, 1$ for the method of Runge, and the Gauß method of order 4. Plot the discretization error and discuss the numerical results.

c) Evaluate the collocation polynomials u of the Gauß method of order 4 applied to the above problem with step size $\tau = 1$ from the intermediate vectors k_i on a sample grid $\Theta_i = i\frac{\tau}{100}$, $i = 0, \ldots, 100$ and plot the discrete trajectories given by the values of u.