Department of Mathematics & Computer Science
Freie Universität Berlin
Prof. Dr. Ralf Kornhuber, Maren-Wanda Wolf

Exercise 4 for the lecture

Numerics II

WS 2011/12

Due: till Thursday, 24. November 12 o'clock

Problem 1 (4 TP)
The discrete flux $\Psi^{t+\tau,t}x(t)$ (approximating $x(t+\tau)$) of Runge-Kutta methods for non-autonomous initial value problems

$$x'(t) = f(t, x(t)), \quad t > t_0, \qquad x(0) = x_0$$

with $f : \Omega \to \mathrm{R}^d$ is specified by

$$\Psi^{t+\tau,t}x = x + \tau \sum_{i=1}^{s} b_i k_i, \qquad\qquad k_i = f(t + c_i\tau, x + \tau \sum_{j=1}^{s} a_{ij}k_j).$$

The coefficients are given in the extended Butcher scheme

$$\begin{array}{c|c} c & \mathcal{A} \\ \hline & b^T \end{array}.$$

A Runge-Kutta method is called invariant with respect to autonomization if the discrete flux coincides with the discrete flux $\overline{\Psi}^\tau$ of the autonomized system

$$y'(t) = F(y(t)), \quad t > 0, \qquad y(0) = (x_0, t_0)^T$$

with $F : \Omega \times \mathrm{R}^+ \to \mathrm{R}^{d+1}$, $y = (x,t)^T \mapsto F(y) = (f(t,x), 1)^T$, i.e.

$$\overline{\Psi}^\tau \begin{pmatrix} x \\ t \end{pmatrix} = \begin{pmatrix} \Psi^{t+\tau,t}x \\ t+\tau \end{pmatrix}.$$

Show that collocation methods are invariant with respect to autonomization.

Problem 2 (4 TP)
Consider the initial value problem

$$x'(t) = \frac{\lambda x}{g(x)}, \qquad t > 0, \qquad x(0) = x_0, \tag{1}$$

with $g(0) > 0$ and $\lambda < 0$ and $g \in C^1(\mathrm{R})$.

a) Show that $x^* = 0$ is an asymptotically stable fixed point of (1).

b) Show that the semi-implicit time discretization

$$\frac{x_{k+1} - x_k}{\tau} = \frac{\lambda x_{k+1}}{g(x_k)}$$

is asymptotically stable, in the sense that $\lim_{k\to\infty} x_k = 0$ for all τ and x_0.

Problem 3 (4 TP)
Prove the following remarks:

a) Let f be dissipativ. Then every fixed point of $x'(t) = f(x)$ is stable.

b) If f satisfies the stronger condition

$$(f(x) - f(y), x - y) \leq \mu |x - y|^2 \qquad \mu < 0,$$

then every fixed point is assymptotically stable.