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Exercise 5 for the lecture

Numerics II

WS 2011/12

Due: till Thursday, 1. November 12 o'clock

Problem 1 (5 TP)
Let $E : \mathrm{R}^n \to \mathrm{R}$ be a convex functional and consider the associated *gradient flow*

$$x'(t) = -\nabla E(x(t)), \qquad x(0) = x_0, \tag{1}$$

where $\nabla E(x(t)) \in \mathrm{R}^n$ is the gradient of E at $x(t)$.

a) Show that $E(x(t)) \leq E(x_0)$ for all $t > 0$. Show then that even $E(x(t)) < E(x_0)$ if $\nabla E(x_0) \neq 0$.

b) Show that $x^* \in \mathrm{R}^n$ is a fixed point of (1) iff (= if and only if) x^* is a minimum of E. Furthermore, show that each isolated fixed point of (1) is stable.

c) Assume that E is strictly convex and coercive. Show that there exists a unique, asymptotically stable fixed point of (1).

Problem 2 (3 TP)
Consider the energy functional

$$E(v) = \frac{1}{2}\int_\Omega (|\nabla v|^2 - bv)dx, \qquad v \in C_0^1(\overline{\Omega}), \quad b \in C(\overline{\Omega}).$$

Show that the gradient of E at $u \in C_0^1(\overline{\Omega})$ is given by

$$\nabla E(u)(v) = (\nabla u, \nabla v) - (b, v), \qquad v \in C_0^1(\overline{\Omega}),$$

where $(\cdot,\cdot)$ denotes the L^2 scalar product.

Problem 3 (4 TP)
Consider the *heat equation*

$$\frac{d}{dt}u(x,t) = \Delta u(x,t) \tag{2}$$

with $u : [a,b] \times \mathrm{R}_0^+ \to \mathrm{R}$, the boundary conditions $u(a,t) = u(b,t) = 0$ and the initial condition $u(x,0) = u_0(x)$. Let there be an equidistant partition $a < x_1 < \ldots < x_n < b$ of the interval $[a,b]$, i.e.,

$$x_i = a + \frac{i(b-a)}{n+1}, \qquad i = 1, \ldots, n.$$

The quantity $h = (b-a)/(n+1)$ is called the *grid size*.

a) Discretize (2) by central difference quotients at the points x_i. Write the spatially discrete problem as

$$u_h'(t) = -A_h u_h(t), \qquad u_h(0) = u_{h,0}$$

with $u_h(t) \in \mathrm{R}^n$ and give $u_{h,0}$ and the matrix A_h.

b) Show that there is a functional $E_h : \mathrm{R}^n \to \mathrm{R}$ such that

$$u_h'(t) = -\nabla E_h(u_h(t)).$$

c) Show that E_h is strictly convex.

Problem 4 (4 PP)
Consider the curve shortening flow given by

$$\begin{aligned} u_t - \frac{1}{|u_x|}\left(\frac{u_x}{|u_x|}\right)_x &= 0 && \text{in } I \times (0,T) \\ u(0,t) &= u(2\pi,t) && \text{in } (0,T) \\ u(\cdot,0) &= u_0 \end{aligned}$$

where $u : I \times (0,T) \to \mathrm{R}^2$ and $u(\cdot,t)$ describes the position of a closed curve in R^2 at the time t parametrized over the interval $I = [0,2\pi]$.
To solve this problem numerically we use the space discrete approximation

$$\begin{aligned} U_j' &= \frac{2}{|U_{j+1}-U_j| + |U_j - U_{j-1}|}\left(\frac{U_{j+1}-U_j}{|U_{j+1}-U_j|} - \frac{U_j - U_{j-1}}{|U_j - U_{j-1}|}\right) \\ U_j &= U_{j+N} && \text{for } j = -1, 0, 1 \end{aligned}$$

with $U : (0,T) \to \mathrm{R}^{2\times N}$. Notice that each U_j is a function, mapping $(0,T)$ to R^2 and approximates the value of $u(x_j,t)$ with the equidistant space grid $(x_j)_{j=0,\ldots,N}$.
Following the ideas for problem 2 on exercise 4 a time discretization is given by

$$\frac{1}{\tau}(U_j^{m+1} - U_j^m) = \frac{2}{(|U_{j+1}^m - U_j^m| + |U_j^m - U_{j-1}^m|)}\left(\frac{U_{j+1}^{m+1} - U_j^{m+1}}{|U_{j+1}^m - U_j^m|} - \frac{U_j^{m+1} - U_{j-1}^{m+1}}{|U_j^m - U_{j-1}^m|}\right).$$

Implement the above iteration in MATLAB as `function [u, t] = CurveShortening(N, tau, T, u0)`, where `N`, `tau`, `T`, and `u0` denote the number of nodes in the space grid, the time step size, the final time and the initial value given as function from I to R^2 respectively. Test your program with interesting initial values and appropriate parameters.