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Exercise 6 for the lecture

Numerics II

WS 2011/12

Due: till Thursday, 8. December 12 o'clock

Problem 1 (4 TP)
Let $\Omega \in \mathrm{R}$ be open. The Newton method for $F \in C^1(\Omega)$ for an initial value $z_0 \in \Omega$ is defined by

$$z_{k+1} = z_k - DF(z_k)^{-1}F(z_k)$$

Prove the following local covergence proposition using Banachs fixed point theorem:

Assume there exists a point $z^* \in \Omega$ with $F(z^*) = 0$, $F \in C^2(\Omega)$ and $DF(z^*) \neq 0$. Then there exists a neighbourhood $U \subset \Omega$ of z^*, such that the Newton method is feasible for every initial value $z_0 \in U$ and $z_k \to z^*$ for $k \to \infty$.

(Hint: Consider the functions $G(z) := z - DF(z)^{-1}F(z)$ and DG.)

Problem 2 (6 PP)
Consider the following nonlinear initial value problem

$$x'(t) = f(x), \qquad t > 0, \qquad x(0) = x_0,$$

with $x(t) \in \mathrm{R}^3$ and

$$f(x) = \begin{pmatrix} -c_1x_1 + c_3x_2x_3 \\ c_1x_1 - c_2x_2^2 - c_3x_2x_3 \\ c_2x_2^2 \end{pmatrix}.$$

a) Implement an implicit Runge-Kutta method for this equation in Matlab as `function [x, t] = RungeKuttaNewton(f, Df, x0, I, tau, b, A, TOL)`, where `f`, `Df`, `x0`, `I`, and `tau` denote the function implementing the right hand side f, the Jacobian Df, the initial value, the time interval, the step size, and the tolerance for the nonlinear solver respectively and the Butcher scheme is given by `b, A`. The returned values `x`, `t` should contain the solution at each time step and the time steps, respectively. Use the simplified Newton method and the stopping criterion presented in the lecture to solve the nonlinear system

$$F(Z) = Z - \tau \begin{pmatrix} \sum_{j=1}^{s} a_{1j} f(x + z_j) \\ \vdots \\ \sum_{j=1}^{s} a_{sj} f(x + z_j) \end{pmatrix} = 0,$$

with $Z = [z_1, \ldots, z_s]^T \in \mathrm{R}^{3s}$.

b) Test your code for the initial value $x_0 = (1, 0, 0)^T$, the time interval $[0, 10]$, and the parameter vector $c = (1, 2, 3)$. Use the implicit Euler method, the implicit midpoint rule and the Gau-method of order 6 for your tests. Estimate the discretization error by computing a sufficiently good reference solution numerically. Plot the estimated error as a function of the step size for all methods and for an appropiate set of step sizes. What convergence orders do you observe ?

c) Plot the total mass $x_1 + x_2 + x_3$ as function of the time. Show that the continuous flux and every Runge-Kutta scheme preserve the total mass.

Problem 3 (5 TP)

A Butcher-scheme $\begin{array}{c|c} & A \\ \hline & b^T \end{array}$ with

$$A = \begin{pmatrix} a_{11} & & & \\ a_{21} & a_{22} & & \\ \vdots & \vdots & \ddots & \\ a_{s1} & a_{s2} & \ldots & a_{ss} \end{pmatrix}$$

defines a *diagonally implicit* Runge-Kutta (DIRK) method Ψ^τ of stage s. If $a_{11} = a_{22} = \ldots = a_{ss}$, Ψ^τ is called a *singly diagonally implicit* Runge-Kutta (SDIRK) method.

a) Describe the advantages of DIRK and SDIRK methods, over usual implicit Runge-Kutta methods.

b) Construct a 2-stage SDIRK method $\Psi^\tau_{2,3}$ of order $p = 3$.

c) Is $\Psi^\tau_{2,3}$ A-stable?