Introduction to Viscosity Solutions for Nonlinear PDEs

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Contents

1 Introduction.

1.1 Partial Differential Equations.

Preliminary facts:

- The theory of viscosity solutions can be applied to study linear and nonlinear Partial Differential Equations of any order.
- A *Partial Differential Equation* (PDE) of order $k \geq 1$ is an equation involving an unknown function u and its derivatives up to the order k. In particular we consider the case $k = 1$ and $k = 2$, i.e.

  $$F(x, u, Du, D^2u) = 0, \quad x \in \Omega \subset \mathbb{R}^n,$$

  where $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{M}_{n,n} \to \mathbb{R}$.
  Notation: $F = F(x, z, p, M)$.
- Note that the unknown function is a scalar function; the theory of viscosity solutions is not in general applied to systems of PDEs, this means that we will not study equations like the Navier-Stokes equations.
- We in general assume that the function u and the PDE (i.e. the function F) are both continuous.
- A PDE of order k is called *linear* if it has the form

  $$Lu = f(x)$$

  where L is a differential operator of order k and L is linear,
  i.e. $L(\lambda u_1 + \mu u_2) = \lambda L(u_1) + \mu L(u_2)$ for any $\lambda, \mu \in \mathbb{R}$.
  Otherwise the PDE is called *nonlinear*.
- E.g. $Du \cdot \eta = f(x)$ is a first-order linear PDE for any $\eta \in \mathbb{R}^n$,
  while $|Du| = f(x)$ is nonlinear.
- E.g. $\Delta u = \sum_{i=1}^n \frac{\partial^2}{\partial x_i \partial x_i} u = 0$ is a second-order linear PDE,
  while $\Delta u + u^2 = 0$ is nonlinear.
- Another very important linear PDE is the heat equation

  $$u_t - \Delta u = 0;$$

  For other examples of linear PDEs see the book of Evans, page 3-4.

- A linear PDE is called *homogeneous* if $f \equiv 0$. In this case, given two solutions u_1 and u_2, then $\lambda u_1 + \mu u_2$ is still a solution for any $\lambda, \mu \in \mathbb{R}^n$.

The theory of viscosity solutions is very useful for studying nonlinear PDEs. Some examples of first- and second-order nonlinear PDEs are the following:

1. *The Eikonal Equation*:
$$|Du| = f(x),$$
which is related to geometric optics (rays).

2. *(Stationary) Hamilton-Jacobi equation*:
$$H(x, u, Du) = 0, \quad \Omega \subset \mathbb{R}^n,$$
where $H : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is called *Hamiltonian* and is continuous and in general convex in p (i.e. in the gradient-variable). The eikonal equation is in particular a (stationary) Hamilton-Jacobi equation.

3. *(Evolution) Hamilton-Jacobi equation*:
$$u_t + H(x, u, Du) = 0, \quad \mathbb{R}^n \times (0, +\infty);$$

4. *The Hamilton-Jacobi-Bellman equation*:
It is a particular Hamilton-Jacobi equation which is very important in control theory and economics.
In this case the Hamiltonian has the form:
$$H(x, p) := \sup_{a \in A} \{-f(x, a) \cdot p - l(x, a)\},$$
where A is a subset of $\mathbb{R}^m$ (or more in general a topological space) and $l : \mathbb{R}^n \times A \to \mathbb{R}$ and $f : \mathbb{R}^n \times A \to \mathbb{R}^n$ are both continuous functions.
For any fixed $\lambda > 0$, the (viscosity) solution of the equation
$$\lambda u + H(x, Du) = 0, \quad x \in \mathbb{R}^n \tag{1}$$
has a particular form. If fact, the solution is known to be "the value function associated to a control problem".
A control function is a measurable function $\alpha : [0, +\infty) \to A$ and a control problem is a nonlinear system of ordinary differential equation:
$$\begin{cases} \dot{y}(t) = f(y(t), \alpha(t)), & t > 0, \\ y(0) = x. \end{cases}$$

We indicate by y_x^α the trajectories solving the previous control system with starting point x.
Given a control function and a corresponding trajectory $y_x^\alpha(t)$, we define a "pay-off" as

$$J(x,\alpha) = \int_0^{+\infty} l(y_x^\alpha(t),\alpha(t))\mathrm{e}^{-\lambda t}dt;$$

the constant $\lambda \geq 0$ is called *interest rate*.
Let $\mathcal{A} := \{\alpha(t) \text{ control}\}$ be the set of all possible controls with value in A. The *value function* of this control problem is given by

$$v(x) = \inf_{\alpha\in\mathcal{A}} J(x,\alpha).$$

Then the function $v(x)$ solves, in the viscosity sense, the equation (1).

5. *Differential Games*: It is a more complicated control problem where two different controls have to be considered (roughly speaking corresponding to the strategies of two different players playing one against the other). Therefore $f = f(x,a,b)$ and $l = l(x,a,b)$ where $a \in A$ and $b \in B$ and A and B are two compact metric spaces (which can be also different). The two set of controls are $\mathcal{A}$ and $\mathcal{B}$ respectively.
Moreover note that in general the two families of controls are not independent (i.e. $\alpha = \alpha[\beta]$), which means that the strategies of each player depend also on the choices of the other player.
Then $\mathcal{A}[\mathcal{B}] \subset \mathcal{A}$ is a set of controls for the player I under suitable restrictions depending on $\beta \in \mathcal{B}$.
Then the *(lower) value function*

$$v(x) = \inf_{\alpha\in\mathcal{A}[\mathcal{B}]} \sup_{\beta\in\mathcal{B}} J(x,\alpha,\beta)$$

solves

$$\lambda u + H(x,Du) = 0, \quad x \in \mathbb{R}^n,$$

where

$$H(x,p) := \min_{b\in B} \max_{a\in A} \{-f(x,a,b)\cdot p - l(x,a,b)\}.$$

6. *The Monge-Ampère equation*:

$$\det(D^2u) = f(x),$$

which has many applications in differential geometry and calculus of variations (Monge-Kantorovitch mass transfer problem).

7. *Evolution by mean curvature flow*:

$$u_t - \Delta u + \left\langle D^2 u \frac{Du}{|Du|}, \frac{Du}{|Du|} \right\rangle = 0.$$

The equation is degenerate elliptic since $F(x, p, M)$ is not well-defined whenever $|p| = 0$.
The equation describes the evolution of a hypersurfaces in the direction of the internal normal and proportional to the mean curvature and it is associated to the gradient-flow of the area-functional, which means that the hypersurface evolves trying to minimize its area.

Other examples of nonlinear PDEs can be found in the book of Evans (page 5).

1.2 Classical and weak solutions.

The two main problems that we are going to consider are:

I *The Dirichlet problem*:

$$\begin{cases} F(x, u, Du, D^2u) = 0, & x \in \Omega, \\ u(x) = g(x), & x \in \partial\Omega, \end{cases} \tag{2}$$

with Ω open and bounded in $\mathbb{R}^n$ and g continuous boundary condition.

II *The Cauchy problem*:

$$\begin{cases} u_t + F(x, u, Du, D^2u) = 0, & (t, x) \in (0, +\infty) \times \mathbb{R}^n, \\ u(0, x) = g(x), & x \in \mathbb{R}^n, \end{cases} \tag{3}$$

where g is a continuous initial condition.

Definition 1.1. Given a PDE of order $k \geq 1$, a function $u : \Omega \to \mathbb{R}^n$ is called *classic solution if* $u \in C^k(\Omega)$ *and* u *solves the PDE at any* $x \in \Omega$.

Remark 1.1. Under suitable assumptions on the PDE (and given suitable initial or boundary conditions), classic solutions are in general unique but they might not exist.

Example 1.1 (Eikonal equation)**.** Let us consider the eikonal equation with $f \equiv 1$ in $\Omega = [-1, 1]$, i.e.

$$|u'(x)| = 1, \quad \text{for } x \in (-1, 1). \tag{4}$$

Let us assume that there exists a classical solution with vanishing Dirichlet condition, which means $u \in C^1(-1,1)$ solving (4) with $u(-1) = u(1) = 0$. By the Mean Value Theorem, there exists a point $\xi \in (-1,1)$ such that $u'(\xi) = 0$, so u cannot solve (4). Hence, since $u \in C^1$ there exists even a non-empty interval $(-a,a)$ (with $0 < a < 1$) such that $|u'(x)| < 1$ for $x \in (-a,a) \subset (-1,1)$, which contradicts (4) in a whole interval.

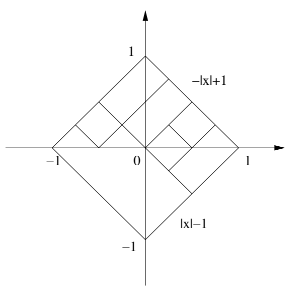
Hence the necessity of weaker notions of solution.
Thinking of the eikonal equation, one could require that the solution is Lipschitz instead of C^1. This implies that the first-derivatives do not exist at any point but just almost everywhere. Then the idea is to require that the equation is satisfied just at the points where the derivatives exist.
So, more in general, one can introduce the following notion.

Definition 1.2 (Almost everywhere solutions)**.** Given a PDE of order $k \geq 1$, a function $u : \Omega \to \mathbb{R}^n$ continuous is called *almost everywhere solution if the derivatives up to the order k exist almost everywhere and u solves the PDE almost everywhere.*

Using again the example of the eikonal equation, we can show that the previous notion is good for existence but very bad for uniqueness.

Example 1.2 (Eikonal equation and Rademacher functions)**.** The function $u(x) = -|x| + 1$ and $v(x) = |x| - 1$ are two different almost everywhere solutions of (4) with vanishing boundary condition $u(-1) = u(1) = 0$. More in general, the Rademacher functions give infinitely many almost everywhere solutions of (4). The Rademacher functions are defined, for any $k \in \mathbb{N}$, as

$$u_k(x) = \begin{cases} x + 1 - \dfrac{i}{2^{k-1}}, \text{ if } x \in \left[-1 + \dfrac{i}{2^{k-1}}, -1 + \dfrac{2i+1}{2^k}\right), \\ -x - 1 + \dfrac{i+1}{2^{k-1}}, \text{ if } x \in \left[-1 + \dfrac{2i+1}{2^k}, -1 + \dfrac{i+1}{2^{k-1}}\right), \end{cases} \quad i = 0, 1, ..., 2^k - 1, \tag{5}$$



Remark 1.2 (Distributional solutions)**.** A different approach to weak solutions for PDEs is given by the so called *distributional solutions*.
These solutions satisfy good properties of existence and uniqueness but they can be applied just if the PDE is linear at least in the derivatives of maximum order.

Crandall and Lions in 1982 introduced a different notion of weak solution which works very well for many first- and second-order nonlinear PDEs, and satisfies properties of existence, uniqueness and stability. Moreover, this new notion selects in a suitable sense the optimal almost everywhere solution.

2 Viscosity solutions.

2.1 Definition and main properties.

Before giving the main definition of this course, we would like to give an intuition where the idea comes from.

Let us suppose to have a classical solution of $-\Delta u = 0$ and consider a function $\varphi \in C^2$, touching u from above at some point x_0, which means $\varphi \geq u$ and $\varphi(x_0) = u(x_0)$; then $u - \varphi$ has a local maximum at the point x_0. Therefore $u - \varphi$ looks locally concave around x_0, which implies

$$0 \geq \Delta(u-\varphi)(x_0) = \Delta u(x_0) - \Delta\varphi(x_0) = -\Delta\varphi(x_0) \;\Rightarrow\; -\Delta\varphi(x_0) \leq 0.$$

Consider a function $\varphi \in C^2$ touching u from below, we can deduce the reverse inequality at the minimum-points, i.e. $-\Delta\varphi(x_0) \geq 0$, at any point x_0 where $u - \varphi$ attends a local minimum.

When the function u is not C^2, we will use the above properties for C^2-functions touching from above and from below to say whenever u solves the PDE in a weak sense.

We can now introduce the definition of viscosity solutions for second-order PDEs:

$$F(x, u, Du, D^2u) = 0, \quad x \in \Omega. \tag{6}$$

Definition 2.1. Let $\Omega \subset \mathbb{R}^n$ be an open set and u continuous in Ω;

(i) We say that u is a *viscosity subsolution* of (6) at a point $x_0 \in \Omega$, if and only if, for any test function $\varphi \in C^2(\Omega)$ such that $u - \varphi$ has a local maximum at x_0, then

$$F(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) \leq 0; \tag{7}$$

(ii) We say that u is a *viscosity supersolution* of (6) at a point $x_0 \in \Omega$, if and only if, for any test function $\varphi \in C^2(\Omega)$ such that $u - \varphi$ has a local minimum at x_0, then

$$F(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) \geq 0; \tag{8}$$

(iii) We say that u is a *viscosity solution* in the open set Ω if u is a viscosity subsolution and a viscosity supersolution, at any point $x_0 \in \Omega$.

Remark 2.1. For PDEs of order k, we can give the same definition requiring a C^k-regularity for the test functions.

Remark 2.2. In order to check the condition for viscosity subsolution (resp. viscosity supersolution), it is enough to require that the function is upper semicontinuous (resp. lower semicontinuous).

Remark 2.3. One can always assume $\varphi(x_0) = u(x_0)$ and, by replacing $\varphi(x)$ with $\varphi(x) + |x - x_0|^4$ (resp. $\varphi(x) - |x - x_0|^4$), that the local minimum (maximum) at x_0 may be assumed strict.

Remark 2.4. Note that the equation $F = 0$ and the equation $-F = 0$ are not equivalent in the viscosity sense (see next example).

Example 2.1 (The eikonal equation)**.** We show that $u(x) = -|x| + 1$ is a viscosity solution of $|u'| - 1 = 0$ in $(-1, 1)$, but it is not of $-|u'| + 1 = 0$.
Let $\varphi \in C^1((-1, 1))$ be such that $u - \varphi$ has a local maximum (resp. minimum) at some $x_0 \in (-1, 1)$. If $x_0 \neq 0$, u is differentiable at x_0 and $|\varphi'(x_0)| = |u'(x_0)| = 1$. The problem is just at the point $x_0 = 0$ where $|x|$ is not differentiable.
So we have just to consider the case where $u - \varphi$ attends a local minimum (resp. maximum) at 0.
Let us first consider the subsolution property, i.e. x_0 local (strict) maximum point and assume $u(0) = \varphi(0)$; then $-|x| \leq u(x) - u(0) \leq \varphi(x) - \varphi(0)$, near 0, which implies

$$1 \geq \frac{\varphi(x) - \varphi(0)}{x}, \quad \text{for } x < 0 \quad \text{and} \quad \frac{\varphi(x) - \varphi(0)}{x} \geq -1, \quad \text{for } x > 0.$$

Since $\varphi \in C^1$, passing to the limit as $x \to 0$, we can conclude $|\varphi'(0)| \leq 1$. Therefore the subsolution condition (7) is satisfied at any point $x_0 \in (-1, 1)$.
To verify the supersolution condition, we assume that $u - \varphi$ attends a local maximum (equal to 0) at the point 0. Proceeding as above we find $D^-\varphi(0) \geq 1$ and $D^+\varphi(0) \leq -1$, which means $D^-\varphi(0) \neq D^+\varphi(0)$.

Therefore there cannot be C^1-functions touching $u(x) = -|x|+1$ from below, which means the condition (8) is trivially verified.
This allows us to conclude that $u(x) = -|x|+1$ is a viscosity solution of (4).

Let us now consider the equation $-|u'(x)| + 1 = 0$ and the function $\varphi(x) = -x^2+1$. It is easy to show that φ is a C^1 function touching u from above at 0. Nevertheless $-|\varphi'(0)| + 1 = 1 > 0$, therefore (7) is not satisfied.

Exercise 2.1. Show that $u(x) = |x|-1$ is a viscosity solution of $-|Du|+1 = 0$ but it is not of $|Du| - 1 = 0$.

Exercise 2.2. Show that, if u is a viscosity solution of (6), then $-u$ solves in the viscosity sense $-F(x, -u, -Du, -D^2u) = 0$.

Viscosity solutions are a good notion in particular for first-order PDEs and second-order elliptic PDEs: this means that they are almost everywhere solutions or classical solutions whenever they are regular enough and moreover the classical solutions are always viscosity solutions, too. As we have seen in the example of the eikonal equation, almost everywhere solutions are instead not always viscosity solutions.

Remark 2.5. Let $u \in C^k$ be viscosity solution of a PDE of order k, then this is a classical solution. To show this, we can just choose as test function u itself in both the subsolution and the supersolution properties.

Proposition 2.1. Let us assume that $u \in C^2$ is a classical solution of (6), then u is a viscosity solution whenever one of the following two is satisfied:

1. The PDE does not depend on D^2u.

2. F satisfies the following assumption:

$$F(x, z, p, M) \leq F(x, z, p, N), \quad M \geq N. \tag{9}$$

Proof. Let $\varphi \in C^2$ be such that $u - \varphi$ has a local maximum at x_0, then $Du(x_0) = D\varphi(x_0)$ and $D^2u(x_0) \leq D^2\varphi(x_0)$.
If (6) is a first-order PDE, then

$$0 = F(x_0, u(x_0), Du(x_0)) = F(x_0, u(x_0), D\varphi(x_0))$$

therefore u is a viscosity subsolution. To check that u is a supersolution is exactly the same.
If (6) is a second-order PDE, we need to assume (9), which implies

$$0 = F(x_0, u(x_0), Du(x_0), D^2u(x_0)) \geq F(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)).$$

If we assume that $u-\varphi$ has a local minimum at x_0, then $D^2u(x_0) \geq D^2\varphi(x_0)$ and so we get the other inequality. □

The previous results say that the notion of viscosity solutions is *consistent* with classical solutions. The property can be also proved at any point where the viscosity solution is differentiable, see e.g Theorem 10.1.1 in the book of Evans, for the first-order case.

Remark 2.6. Assumption (9) tells that the equation is *degenerate elliptic.* The main example is $-\Delta u = f(x)$. In that case

$$F(x, z, p, M) = -\mathrm{Tr}(M) - f(x).$$

Remark 2.7. For second-order PDEs, we usually assume also that

$$F(x, z, p, M) \leq F(x, z', p, M), \quad z \leq z'. \tag{10}$$

In this case the function F is said *proper.* An example of a PDE not satisfying assumption (10), is $F(x, z, p, M) = b(z)p - f(x)$ in $\mathbb{R}$ which is never increasing neither decreasing in z (e.g. one can take $p = +1$ and $p = -1$). Hence we do not consider PDEs like $b(u)u_x = f(x)$.
Condition (10) is not necessary for consistence but it is for comparison principles (i.e. uniqueness), which are one of the main properties of viscosity solutions. Without assumption (10), comparisons do not hold even for classical solutions. Note that comparison principles for classical solutions ensure that the definition of viscosity solutions selects the correct solution.

Next we show an other key-property for viscosity solutions: *stability.*
Stability is key for solutions of PDEs since PDEs usually describe models in physics, economics, etc. and therefore the data of the equation (boundary and initial conditions but also the coefficient of F) come from measurement, hence they are never "exact".

Proposition 2.2. *Let F_ε and F be continuous in all the variables and such that $F_\varepsilon \to F$, as $\varepsilon \to 0^+$ and let u_ε be a viscosity solutions of*

$$F_\varepsilon(x, u_\varepsilon, Du_\varepsilon, Du^2_\varepsilon) = 0$$

such that $u_\varepsilon \to u$ (locally uniformly), as $\varepsilon \to 0^+$.
Then u is a viscosity solution of

$$F(x, u, Du, D^2u) = 0.$$

Proof. First note that since the convergence is uniform, then u is continuous. Let $\varphi \in C^2$ be such that $u - \varphi$ has a local strict maximum at x_0 and let us denote by $B_R(x_0)$ the ball where $u(x) - \varphi(x) \leq u(x_0) - \varphi(x_0)$. Consider $K = \overline{B}_{\frac{R}{2}}(x_0)$ and let x_ε be the maximum point of $u_\varepsilon - \varphi$ in K.

We want to show that $x_\varepsilon \to x_0$, as $\varepsilon \to 0^+$.
Since $x_\varepsilon \in K$ compact set, (up to a subsequence) there exists $y \in K$ such that $x_\varepsilon \to y$. Note that x_ε is a maximum point in K so in particular

$$u_\varepsilon(x_0) - \varphi(x_0) \leq u_\varepsilon(x_\varepsilon) - \varphi(x_\varepsilon).$$

Passing to the limit as $\varepsilon \to 0^+$, we get

$$u(x_0) - \varphi(x_0) \leq u(y) - \varphi(y).$$

Therefore since x_0 is a strict local maximum, we deduce $x_0 = y$.
Now to conclude is very easy. In fact, u_ε is a viscosity solution, so

$$F_\varepsilon(x_\varepsilon, u_\varepsilon(x_\varepsilon), D\varphi(x_\varepsilon), D\varphi^2(x_\varepsilon)) = 0$$

Since $\varphi \in C^2$ and F_ε is continuos, passing to the limit as $\varepsilon \to 0^+$, we get

$$F(x_0, u(x_0), D\varphi(x_0), D\varphi^2(x_0)) = 0;$$

therefore u is a viscosity subsolution.
To show that u is a viscosity supersolution, one proceeds similarly. □

The next remark gives an application on the previous stability result to the existence problem. This method is at the origin of the name "viscosity solution".

Remark 2.8 (Existence for Hamilton-Jacobi equations by viscosity approximations)**.** Let us consider the first-order Cauchy problem:

$$\begin{cases} u_t + H(x, Du) = 0, \\ u(0) = g. \end{cases} \tag{11}$$

A method for solving the problem is to add a "viscosity term" to the equation, i.e. $-\varepsilon \Delta u$, for any $\varepsilon > 0$. In this way, the equation becomes a heat equation plus a nonlinear lower-order term:

$$\begin{cases} u^\varepsilon_t - \varepsilon \Delta u^\varepsilon + H(x, Du^\varepsilon) = 0, \\ u^\varepsilon(0) = g. \end{cases} \tag{12}$$

Existence for equation (12) is easier to prove and, since the equation is parabolic and linear in the high-order operator, it is in general possible to show the existence of classical solutions u^ε. Then we need just to get an estimate for the solutions, uniform in $0 < \varepsilon << 1$ and, by Ascoli-Arzela Theorem, we can define

$$u(t,x) = \lim_{\varepsilon \to 0^+} u^\varepsilon(t,x),$$

where the above limit is locally uniform.
Therefore using the stability-result for viscosity solutions (Proposition 2.2), $u(t,x)$ solves, in the viscosity sense, the limit-equation, which is the Cauchy problem (11).

The same method can be applied to the Dirichlet problem.
More information about the vanishing viscosity method can be found in the book of Evans (Section 10.1) and in the following papers: P.L. Lions, *Generalized Solutions of Hamilton-Jacobi Equations* Research, Notes in Mathematics 69, 1982 (Section 1.4 and Section 8); M.G. Crandall, L.C. Evans and P.L. Lions, *Some properties of viscosity solutions of Hamilton-Jacobi equations*, Trans. Am. Mat. Soc. 282, 1984 (Theorem 3.1.); G. Barles and B. Perthame, *Exit time problems in optimal control and vanishing viscosity method*, SIAM J. Control Opt. 26, 1988; and many others.

Now we show the nice behavior of viscosity solutions w.r.t. the operations of infimum and supremum.

Proposition 2.3. Let $v \in \mathcal{F}$ be a family of viscosity subsolutions (resp. viscosity supersolutions) of (6) and look at

$$u(x) = \sup_{v \in \mathcal{F}} v(x) \quad (\text{resp. } u(x) = \inf_{v \in \mathcal{F}} v(x)).$$

For sake of simplicity, we also assume that $u(x)$ is upper semicontinuous (resp. lower semicontinuous), then u is a viscosity subsolution (resp. viscosity supersolution) of (6).

Proof. We show just the property for the infimum.
Let $\varphi \in C^2$ be such that $u - \varphi$ has a local strict minimum at x_0, i.e. there exists $R > 0$ such that

$$u(x_0) - \varphi(x_0) < u(x) - \varphi(x), \quad \forall\, x \in B_r(x_0) \backslash \{x_0\}.$$

Since $u(x_0) = \inf\{v(x_0) \,|\, v \in \mathcal{F}\}$, for any $n \in \mathbb{N}$, there exists $v_n \in \mathcal{F}$ such that

$$v_n(x_0) < u(x_0) + \frac{1}{n}.$$

Let x_n be a sequence of points where $v_n - \varphi$ attend minimum in $\overline{B}_{\frac{R}{2}}(x_0)$, i.e.

$$v_n(x_n) - \varphi(x_n) \leq v_n(x) - \varphi(x), \quad \forall\, x \in K := \overline{B}_r(x_0). \tag{13}$$

Up to a subsequence, $x_n \to y \in K$. Using that $u \leq v_n$, for any $v_n \in \mathcal{F}$, inequality (13) at x_0 gives

$$u(x_n) - \varphi(x_n) \leq v_n(x_n) - \varphi(x_n) \leq v_n(x_0) - \varphi(x_0) < u(x_0) + \frac{1}{n} - \varphi(x_0).$$

Passing to the liminf, as $n \to +\infty$, and using the lower semicontinuity of u and the continuity of φ, we get

$$u(y) - \varphi(y) \leq u(x_0) - \varphi(x_0).$$

Exactly as in the proof of the stability, the assumption that x_0 is a strict minimum point allows us to conclude that $x_0 = y$.
Therefore, since v_n are viscosity supersolutions, we know that

$$F(x_n, \varphi(x_n), D\varphi(x_n), D^2\varphi(x_n)) \geq 0,$$

which, passing to the limit as $n \to +\infty$, proves that u is viscosity supersolution of equation (6). □

Remark 2.9. Note that in general the supremum of viscosity supersolutions (resp. the infimum of viscosity subsolutions) is not a viscosity supersolution (resp. viscosity subsolution). Moreover it is not necessary to assume the right semicontinuity for the supremum or the infimum. Nevertheless without this assumption the result is still true but one has to use the theory of discontinuous viscosity solution, which means to deal with the lower and the upper semicontinuous envelopes.

As application of the behavior w.r.t. the operations of infimum and supremum, we introduce a first method for existence of viscosity solutions.

2.2 Existence by Perron's method

In 1987 H. Ishii used for the first time the *Perron's method* to solve nonlinear first-order equations (*Perron's method for Hamilton-Jacobi equations*, Duke Math. J. 55). This method had been introduced in 1923 by Oskar Perron in order to find solutions for the Laplace equation and consists in building a solution as the supremum of a suitable family of viscosity subsolutions. Since the supremum of viscosity subsolutions is a viscosity subsolution, one has just to prove that it is a viscosity supersolution, too.
The Perron's method can be sketched, as follows:

Theorem 2.1 (Perron's Method). *Let us assume that*

1. *Comparison principle for* (6) *holds, i.e. given u viscosity subsolution and v viscosity supersolution satisfying the same boundary condition, then $u \leq v$.*

2. *Suppose that there exist $\underline{u}$ and $\overline{u}$ which are, respectively, a viscosity subsolution and a viscosity supersolution, satisfying the same boundary condition.*

We define

$$W(x) = \sup\{w(x) \,|\, \underline{u} \leq w \leq \overline{u} \text{ and } w \text{ viscosity subsolution}\},$$

Then W is a viscosity solution of (6), *which satisfies the same boundary condition satisfied by $\underline{u}$ and $\overline{u}$.*

Let us set

$$\mathcal{F} = \{w \,|\, \underline{u} \leq w \leq \overline{u} \text{ and } w \text{ viscosity subsolution}\},$$

so that W can be written as

$$W(x) = \sup_{w \in \mathcal{F}} w(x).$$

To prove the Perron's Method, we need to use the following result.

Lemma 2.1. If $v \in \mathcal{F}$ is not a viscosity supersolution, then there exists $w \in \mathcal{F}$ such that $v(y) < w(y)$ at some $y \in \Omega$.

Proof. Let us assume $v \in \mathcal{F}$ is not a supersolution, this means that there exists $y_0 \in \Omega$ and $\varphi \in C^2$ such that $v - \varphi$ has a local minimum at y_0 and

$$F(y_0, v(y_0), D\varphi(y_0), D^2\varphi(y_0)) < -\eta, \tag{14}$$

for some $\eta > 0$.
We may assume that $\varphi(y_0) = v(y_0)$ and that the local minimum at y_0 is strict. We are going to use the test function φ to build a subsolution $w \in \mathcal{F}$ such that $v(y_0) < w(y_0)$.
First note that $v(y_0) < \overline{u}(y_0)$. In fact, if it would be $v(y_0) = \overline{u}(y_0)$, recalling that $\overline{u} \geq v \geq \varphi$, then $\overline{u} - \varphi$ would have a local minimum at y_0. Since $\overline{u}$ is a viscosity supersolution, that would mean $F(y_0, v(y_0), D\varphi(y_0), D^2\varphi(y_0)) \geq 0$, which contradicts (14).
Therefore we can find $\delta_1 > 0$ such that $v(y) + \delta_1 < \overline{u}(y)$, on $B_{\delta_1}(y_0)$.

Now we use the continuity of F and (14), which implies that there exists δ_2 such that, whenever x, u, p, M satisfy

$$|x-y_0| < \delta_2,\ |u(y_0)-v(y_0)| < \delta_2,\ \|p - D\varphi(y_0)\| < \delta_2,\ \left\|M - D^2\varphi(y_0)\right\| < \delta_2,$$

then $F(x, u, p, M) < 0$. Let be $\delta := \min\{\delta_1, \delta_2\} > 0$; we define

$$w(x) = \begin{cases} \max\{\phi(x) + \delta, v(x)\}, & x \in B_\delta(y_0), \\ v(x), & x \in \Omega \backslash B_\delta(y_0). \end{cases}$$

Then w is continuous and is a viscosity subsolution. Moreover $\underline{u} \leq w \leq \overline{u}$, then $w \in \mathcal{F}$ and $w(y_0) = \varphi(y_0) + \delta > v(y_0)$ so the lemma is proved. □

Proof of Theorem 2.1. Note that $\mathcal{F} \neq \emptyset$, therefore W is a viscosity subsolution of (6) satisfying the same boundary condition (since $g(x) = \underline{u}(x) \leq w(x) \leq \overline{u}(x) = g(x)$ on $x \in \partial\Omega$).
The fact that W is a viscosity supersolution follows by Lemma 2.1, assuming that W is continuous. In fact, if we assume that W is not a viscosity supersolution, then the lemma contradicts the fact that W is defined as the supremum on $\mathcal{F}$, at any point.
Nevertheless, as we have already remarked, in general W is not continuous but just lower semicontinuous.
However the proof still works substituting W and the supersolutions used to prove the lemma by their semicontinuous envelopes. □

Perron's method is good to prove existence-results for very general PDEs (both first-order and second-order cases). The limit of this method is that does not give representative formula for the solution neither information on the regularity of the solution.
For the first-order case, we will see later an other method which gives much more information on the solution (called Hopf-Lax formula).

2.3 Further properties.

Let us conclude this introduction on the theory of viscosity solutions, showing the following properties.

Proposition 2.4. Let us consider $\Phi \in C^2(\mathbb{R})$, with $\Phi' > 0$ and $\Phi(\mathbb{R}) = \mathbb{R}$, if u is a viscosity subsolution (resp. viscosity supersolution) of the equation (6),
then $v := \Phi \circ u$ is a viscosity subsolution (resp. viscosity supersolution) of

$$F(x, \Psi(v(x)), \Psi'(v(x))Dv, \Psi''(v(x))DvDv^T + \Psi'(v(x))D^2v) = 0, \qquad (15)$$

with $\Psi = \Phi^{-1}$.

Proof. Note that if $\Phi' > 0 \implies \Psi' > 0$; therefore Ψ is increasing.
As usual, we show only one property, i.e. that v is a viscosity subsolution of (15), since the other one is exactly the same.
Let $\varphi \in C^2$ be such that $v - \varphi$ has a local maximum at x_0, that we can assume $= 0$. Then

$$v(x) \leq \varphi(x), \quad \text{near } x_0 \quad \text{and} \quad v(x_0) = \varphi(x_0). \tag{16}$$

Since Ψ is increasing, then (16) implies

$$u(x) \leq \Psi \circ \varphi(x), \quad \text{near } x_0 \quad \text{and} \quad u(x_0) = \Psi \circ \varphi(x_0).$$

Therefore, $\widetilde{\phi} := \Psi \circ \varphi \in C^2$ is a test function touching u from above at x_0, i.e.

$$F(x_0, u(x_0), D\widetilde{\phi}(x_0), D^2\widetilde{\phi}(x_0)) \leq 0.$$

To conclude it is sufficient to observe that $D\widetilde{\phi}(x_0) = \Psi'(\varphi(x_0))D\varphi(x_0)$ while $D^2\widetilde{\phi}(x_0) = \Psi''(\varphi(x_0))D\varphi(x_0)D\varphi^T(x_0) + \Psi'(\varphi(x_0))D^2\varphi(x_0)$ and $v(x_0) = \varphi(x_0)$. $\square$

Exercise 2.3. *What happens if instead $\Phi' < 0$?*

Now we want to show what happens when the transformation is a change of charts, i.e. let us consider $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ diffeomorphism smooth enough.
For sake of simplicity, we consider just the first-order case, i.e.

$$F(x, z, p, M) = F(x, z, p).$$

Note that in this case $D\Phi$ is not a vector but a $n \times n$ symmetric matrix, (which is usually called Jacobian-matrix and indicate by $J\Phi$.)

Proposition 2.5. Let u be a viscosity solution of

$$F(x, u, Du) = 0,$$

and $\Phi \in C^1$ diffeomorphism. Then $v := u \circ \Phi$ is a viscosity solution of

$$F\big(y, v\big(\Phi^{-1}(y)\big), (D\Phi^{-1})^T(y)\, Dv\big(\Phi^{-1}(y)\big)\big) = 0. \tag{17}$$

Proof. We show just that v is a viscosity subsolution of (17). Hence, let $\varphi \in C^1$ be such that $v - \varphi$ has a local maximum at x_0; this means

$$u(\Phi(x)) - \varphi(x) \leq u(\Phi(x_0)) - \varphi(x_0) \implies u(y) - \varphi \circ \Phi^{-1}(y) \leq u(y_0) - \varphi \circ \Phi^{-1}(y_0)$$

with $y = \Phi(x)$ and $y_0 = \Phi(x_0)$.
Therefore $\widetilde{\varphi} := \varphi \circ \Phi^{-1} \in C^1$ is a test function touching u from above at y_0. Applying the viscosity subsolution property for u and remarking that $u = v \circ \Phi^{-1}$ and $D\widetilde{\varphi}(y) = (D\Phi^{-1})^T(y)\, D\varphi(\Phi^{-1}(y))$, we can conclude. $\square$

Remark 2.10. Note that we do not need to assume any condition on the diffeomorphism Φ (but the necessary regularity).

Remark 2.11. A similar property holds also for second-order PDEs but the formula is much more complicate. The idea is always to derive the equation heuristically assuming everything smooth. Then, if you have a diffeomorphism smooth enough, from the result for classical solutions the corresponding one for viscosity solutions follows trivially.

Exercise 2.4. (Difficult) Assuming $n = 1$, state the corresponding of Proposition 2.5 for the second-order case.

Exercise 2.5. (Very Difficult: need to know tensorial calculus) Do the previous exercise with $n > 1$ (note that in this case $D^2\Phi$ is a 3-tensor) .

3 Control Systems and Hamilton-Jacobi-Bellman Equations.

Let us introduce the notations:

- $x \in \mathbb{R}^n$ and $t \geq 0$;
- $A \subset \mathbb{R}^m$ closed and bounded (e.g. $A = \overline{B}_1(0)$), hence compact;
- A *control* $\alpha(t)$ is a function $\alpha : [0, +\infty) \to A$ measurable;
- $\mathcal{A}$ is the set of all the controls;
- The *dynamics* is a continuous function $f : \mathbb{R}^n \times A \to \mathbb{R}^n$, which we assume to be Lipschitz in x, uniformly in $a \in A$;

We consider the following *control system* (called also *state equation*):

$$\begin{cases} \dot{y}(t) = f(y(t), \alpha(t)), \quad t > 0, \\ y(0) = x. \end{cases} \tag{18}$$

Since f is Lipschitz and the control is measurable then, for any $\alpha \in \mathcal{A}$ there exists a unique solution of (18), that we indicate by y_x^α, given by

$$y_x^\alpha(t) = x + \int_0^t f(y_x^\alpha(s), \alpha(s))ds, \quad \forall\, t > 0,\ x \in \mathbb{R}^n,\ \alpha \in \mathcal{A}.$$

Moreover the following properties hold:

(i) $y_x^\alpha(t)$ is absolutely continuous on the compact sets of $[0,+\infty)$;

(ii) $|y_x^\alpha(t) - y_z^\alpha(t)| \le e^{Lt}|x-z|$, for all $t>0$, $x,z\in\mathbb{R}^n$ and $\alpha\in\mathcal{A}$, where by L we indicate the Lipschitz constant of the dynamics f;

(iii) $|y_x^\alpha(t)| \le \big(|x| + \sqrt{2Kt}\big)e^{Kt}$, for all $\alpha\in\mathcal{A}$ and $t>0$, where $K := L + \sup\{|f(0,a)|\,|\, a\in A\}$.
(Note that $K<+\infty$ since A compact and f continuous).

(iv) $|y_x^\alpha(t) - x| \le M_x\, t$, for all $\alpha\in\mathcal{A}$ and $t\in[0,1/M_x]$,
where $M_x := \sup\{|f(y,a)|\,|\,|y-x|\le 1,\ a\in A\}$.

Note that the property (i) implies that $y_x^\alpha(t)$ satisfies (18) almost everywhere for $t\in[0,+\infty)$, for any $x\in\mathbb{R}^n$ and $\alpha\in\mathcal{A}$, the property (ii) means that the trajectories $y_x^\alpha(t)$ are Lipschitz in x, uniformly w.r.t to $\alpha\in\mathcal{A}$ and and locally uniformly in time, while the property (iii) tells that the trajectories $y_x^\alpha(t)$ are bounded in any compact interval $[0,T]$, uniformly in $\alpha\in\mathcal{A}$ and locally uniformly in x.

For more details existence (local and global) and uniqueness for general ODEs, one can look at T.W. Körner, *A companion to analysis* (2004) and F. Tricomi, *Differential Equations* (1961) and many others text-books on ODEs. For more information on the solutions of (18) and in particular on the properties (i), (ii) and (iii) for $y_x^\alpha(t)$, we refer to the book of Bardi and Capuzzo Dolcetta, Section III.5.

Remark 3.1 (Uniform continuity)**.** A continuous function $h:\Omega\to\mathbb{R}^m$ is *uniformly continuous* in $\Omega\subset\mathbb{R}^n$ if there exists $\omega(r)$ modulus of continuity (i.e. $\omega:[0,+\infty)\to[0,+\infty)$ continuous, nondecreasing with $\omega(0)=0$) such that

$$|h(x)-h(y)| \le \omega(|x-y|), \quad \forall\, x,y\in\Omega.$$

The uniform continuity is weaker than the Lipschitz continuity.
In general, we assume Lipschitz regularity for the functions involved in the control problem. Nevertheless, it is possible to prove the same results, requiring that such functions are just uniformly continuous, instead of Lipschitz. The proofs are the same, replacing $L|x-y|$ by the modulus of continuity $\omega(|x-y|)$ and using the relative properties.

Let us introduce a *running cost* which is a function $l:\mathbb{R}^n\times A\to\mathbb{R}$ and a *terminal cost* $g:\mathbb{R}^n\to\mathbb{R}$ and let us assume that both the running cost and the terminal cost are Lipschitz and bounded in x, uniformly w.r.t. $a\in A$.

The *cost functional* (called also *pay-off*) is given by

$$J(t,x;\alpha) := \int_0^t \mathrm{e}^{-\lambda s}\, l(y_x^\alpha(s),\alpha(s))ds + \mathrm{e}^{-\lambda t} g(y_x^\alpha(t)),$$

where $\lambda \geq 0$ is a constant called *interest rate*.

The aim is to minimize $J(t,x;\alpha)$ and if it is possible to find a control which realize such a minimum. Therefore we are interested in studying the following function:

$$v(t,x) := \inf_{\alpha\in\mathcal{A}} J(t,x;\alpha), \tag{19}$$

which is called *value function*.

Remark 3.2. *A second important question is to find conditions ensuring the existence of an* optimal control, *i.e. some* $\overline{\alpha} \in \mathcal{A}$ *such that* $J(t,x;\overline{\alpha}) = v(t,x)$. *Note that, whenever an optimal control exists, then*

$$v(t,x) = \min_{\alpha\in\mathcal{A}} J(t,x;\alpha).$$

Example 3.1. *Let us consider the easy case where* $\mathcal{A} = \overline{B}_1(0)$, $f(x,a) = a$, *therefore*

$$y_x^\alpha(t) = x + \int_0^t \alpha(s)ds.$$

To define the pay-off we assume $\lambda = 0$, $l \equiv 0$ *and* $g(x) = |x|$, *i.e.*

$$J(t,x;\alpha) = |y_x^\alpha(t)| = \left| x + \int_0^t \alpha(s)ds \right|.$$

We want to minimize J, *therefore we want that, at the time* $t > 0$, *the point* $y(t) = x + \int_0^t \alpha(s)ds$ *is as close to 0 as possible. The idea is to go as fast as possible to 0 and, once there, to stop. Recall that* $|\alpha| \leq 1$, *this means to choose as control* $\alpha(t) = -\frac{x}{|x|}$, *if* $t \leq |x|$ *and* $\alpha(t) = 0$, *if* $t > |x|$, *which gives* $J(t,x;\alpha) = \left(1 - \frac{t}{|x|}\right)|x|$, *if* $t \leq |x|$, *and* $J(t,x;\alpha) = 0$, *if* $t > |x|$. *Therefore the value function has the expression*

$$v(t,x) = \max\left\{ \left(1 - \frac{t}{|x|}\right)|x|, 0 \right\}.$$

Proposition 3.1. *Under the above assumptions on the functions* f, l, g, *then*

1. v *is bounded and continuous on* $[0,T] \times \mathbb{R}^n$*;*

2. *Assuming $\lambda > 0$, v is bounded and continuous on $[0, +\infty) \times \mathbb{R}^n$.*

Proof. Since $v(\tau, z) = \inf_{\alpha \in \mathcal{A}} J(\tau, z; \alpha)$, for any $\varepsilon > 0$, there exists $\overline{\alpha}_\varepsilon \in \mathcal{A}$ such that

$$v(\tau, z) \geq J(\tau, z; \overline{\alpha}_\varepsilon) - \varepsilon.$$

We are going to use the control $\overline{\alpha}_\varepsilon$ to estimate $v(t, x) - v(\tau, z)$; in fact:

$$v(t,x) - v(\tau, z) \leq J(t, x; \overline{\alpha}_\varepsilon) - J(\tau, z; \overline{\alpha}_\varepsilon) + \varepsilon = \int_0^t \mathrm{e}^{-\lambda s}\, l(y_x^{\overline{\alpha}_\varepsilon}(s), \overline{\alpha}_\varepsilon(s))ds$$
$$- \int_0^\tau \mathrm{e}^{-\lambda s}\, l(y_z^{\overline{\alpha}_\varepsilon}(s), \overline{\alpha}_\varepsilon(s))ds + \mathrm{e}^{-\lambda t} g(y_x^{\overline{\alpha}_\varepsilon}(t)) - \mathrm{e}^{-\lambda \tau} g(y_z^{\overline{\alpha}_\varepsilon}(\tau)) + \varepsilon.$$

Let us assume $0 \leq \tau \leq t \leq T$, then the above inequality becomes:

$$v(t,x) - v(\tau, z) \leq \int_0^\tau \mathrm{e}^{-\lambda s} \left| l(y_x^{\overline{\alpha}_\varepsilon}(s), \overline{\alpha}_\varepsilon(s)) - \ l(y_z^{\overline{\alpha}_\varepsilon}(s), \overline{\alpha}_\varepsilon(s)) \right| ds$$
$$+ \int_\tau^t \mathrm{e}^{-\lambda s}\, l(y_x^{\overline{\alpha}_\varepsilon}(s), \overline{\alpha}_\varepsilon(s))ds + \mathrm{e}^{-\lambda \tau} \left| g(y_x^{\overline{\alpha}_\varepsilon}(t)) - g(y_z^{\overline{\alpha}_\varepsilon}(\tau)) \right| + \varepsilon$$
$$\leq \mathrm{Lip}(l) \mathrm{e}^{L\,T} |x - z| \int_0^\tau \mathrm{e}^{-\lambda s} ds + M \int_\tau^t \mathrm{e}^{-\lambda s} ds$$
$$+ \mathrm{e}^{-\lambda \tau} \mathrm{Lip}(g) \left| y_x^{\overline{\alpha}_\varepsilon}(t) - y_z^{\overline{\alpha}_\varepsilon}(\tau) \right| + \varepsilon,$$

where we have used the property (ii) for the solutions of (18) and $L = \mathrm{Lip}(f)$ is the Lipschitz constant of f while $\mathrm{Lip}(l)$ and $\mathrm{Lip}(g)$ are respectively the Lipschitz constant of l and g and M is a global bound of l.

Note that, since $\lambda, s \geq 0$, then $\mathrm{e}^{-\lambda \tau} \leq 1$, and moreover $\tau < T$, that means

$$v(t,x) - v(\tau, z) \leq \mathrm{Lip}(l) \mathrm{e}^{L\,T} T |x - z| + M|t - \tau| + \mathrm{Lip}(g) \left| y_x^{\overline{\alpha}_\varepsilon}(t) - y_z^{\overline{\alpha}_\varepsilon}(\tau) \right| + \varepsilon.$$

We rest to estimate $\left| y_z^{\overline{\alpha}_\varepsilon}(\tau) - y_x^{\overline{\alpha}_\varepsilon}(t) \right|$, using the properties given for $y_x^\alpha(t)$. By triangle inequality:

$$\left| y_z^{\overline{\alpha}_\varepsilon}(\tau) - y_x^{\overline{\alpha}_\varepsilon}(t) \right| \leq \left| y_z^{\overline{\alpha}_\varepsilon}(\tau) - y_x^{\overline{\alpha}_\varepsilon}(\tau) \right| + \left| y_x^{\overline{\alpha}_\varepsilon}(\tau) - y_x^{\overline{\alpha}_\varepsilon}(t) \right|,$$

Property (i) implies

$$\left| y_z^{\overline{\alpha}_\varepsilon}(\tau) - y_x^{\overline{\alpha}_\varepsilon}(\tau) \right| \leq \mathrm{e}^{LT} |x - z|;$$

moreover

$$\left| y_x^{\overline{\alpha}_\varepsilon}(\tau) - y_x^{\overline{\alpha}_\varepsilon}(t) \right| = \left| \int_\tau^t f(y_x^{\overline{\alpha}_\varepsilon}(s), \overline{\alpha}_\varepsilon(s))ds \right| \leq \int_\tau^t |f(y_x^{\overline{\alpha}_\varepsilon}(s), \overline{\alpha}_\varepsilon(s))| ds \leq C|t - \tau|,$$

since f is continuous and, by property (iii), $y_x^{\overline{\alpha}_\varepsilon}(t)$ is bounded in compact intervals, uniformly in $\alpha \in \mathcal{A}$.

Hence

$$\left|y_z^{\overline{\alpha}_\varepsilon}(\tau) - y_x^{\overline{\alpha}_\varepsilon}(t)\right| \leq \mathrm{e}^{LT}|x-z| + C|t-\tau|,$$

where L and C are constants, independent of $\overline{\alpha}_\varepsilon$.
Then we can sum up the following estimate:

$$v(t,x) - v(\tau,z) \leq \big(\mathrm{Lip}(l)T + \mathrm{Lip}(g)\big)\mathrm{e}^{L\,T}|x-z| + \big(M + C\,\mathrm{Lip}(g)\big)|t-\tau| + \varepsilon,$$

where all the constants at the right-hand side do not depend on $\overline{\alpha}_\varepsilon$.
Passing to the limit as $\varepsilon \to 0^+$ and swapping x and z and, τ and t, we can conclude:

$$|v(t,x) - v(\tau,z)| \leq \mathrm{e}^{L\,T}(\mathrm{Lip}(l)T + \mathrm{Lip}(g))|x-z| + (M + C\,\mathrm{Lip}(g))|\tau - t| \to 0,$$

as $|x-z| \to 0$ and $|\tau - t| \to 0$; hence the continuity of $v(t,x)$ in t and x.

To prove that $v(t,x)$ is bounded is sufficient to remark that

$$|J(t,x;\alpha)| = \left|\int_0^t \mathrm{e}^{-\lambda s}\, l(y_x^\alpha(s),\alpha(s))ds + \mathrm{e}^{-\lambda t} g(y_x^\alpha(t))\right| \leq M\int_0^t \mathrm{e}^{-\lambda s} ds + G,$$

for any $t \geq 0$, $x \in \mathbb{R}^n$ and $\alpha \in \mathcal{A}$, where M and G are, respectively, the smallest constant bounding l and g.
Then, whenever $t \in [0,T]$,

$$|J(t,x;\alpha)| \leq MT + G < +\infty,$$

for any $\lambda \geq 0$.
In the case $T = +\infty$, we have to assume $\lambda > 0$, so that

$$|J(t,x;\alpha)| \leq \frac{M}{\lambda}\big(1 - \mathrm{e}^{-T}\big) + G \leq \frac{M}{\lambda} + G < +\infty,$$

which proves the result in 2. and conclude the proposition.
Note: whenever $\lambda = 0$, the above estimate for $|J(t,x;\alpha)|$ explodes to $+\infty$.
□

Remark 3.3. In the previous proposition we have indeed showed that the value function is Lipschitz in t (uniformly w.r.t. the space) and in x (locally uniformly w.r.t. the time).

Remark 3.4. It is trivial to see that $v(x,0) = g(x)$, for all $x \in \mathbb{R}^n$, which means that the value function $v(t,x)$ assumes g as initial datum.

The *Dynamic Programming Principle* is one of the key-properties for value functions and it follows by the *semigroup property* for the trajectories of (18), that means, for any $0 \leq \tau \leq t$ and for any $x \in \mathbb{R}^n$,

$$y_x^\alpha(t) = y_y^{\widetilde{\alpha}}(t-\tau),$$

where $y := y_x^\alpha(\tau)$, and $\widetilde{\alpha}(s) := \alpha(s+\tau)$;
(so that $\widetilde{\alpha}(0) = \alpha(\tau)$ and $\widetilde{\alpha}(t-\tau) = \alpha(t)$).

Proposition 3.2 (Dynamic Programming Principle)**.** For all $x \in \mathbb{R}^n$ and $0 \leq \tau \leq t$, it holds

$$v(t,x) = \inf_{\alpha \in \mathcal{A}} \left\{ \int_0^\tau \mathrm{e}^{-\lambda s} l(y_x^\alpha(s), \alpha(s)) ds + \mathrm{e}^{-\lambda \tau} v(t-\tau, y_x^\alpha(\tau)) \right\}. \tag{20}$$

Proof. If $t = 0$ there is nothing to prove. Moreover, if $\tau = t$, the the identity follows trivially by Remark 3.4.
Let us fix $0 < \tau < t$ and $\alpha \in \mathcal{A}$ and define $\widetilde{\alpha}(s) := \alpha(s+\tau)$, then

$$J(t,x;\alpha) = \int_0^t \mathrm{e}^{-\lambda s}\, l(y_x^\alpha(s), \alpha(s)) ds + \mathrm{e}^{-\lambda t} g(y_x^\alpha(t))$$
$$= \int_0^\tau \mathrm{e}^{-\lambda s}\, l(y_x^\alpha(s), \alpha(s)) ds + \int_\tau^t \mathrm{e}^{-\lambda s}\, l(y_x^\alpha(s), \alpha(s)) ds + \mathrm{e}^{-\lambda \tau} \mathrm{e}^{-\lambda (t-\tau)} g(y_x^\alpha(t)).$$

Now we make a change of variables in the second integral: $s \mapsto \widetilde{s} = s - \tau$, which gives

$$\begin{aligned} J(t,x;\alpha) &= \int_0^\tau \mathrm{e}^{-\lambda s}\, l(y_x^\alpha(s), \alpha(s)) ds + \mathrm{e}^{-\lambda \tau} J(t-\tau, y_x^\alpha(\tau); \widetilde{\alpha}) \\ &\geq \int_0^\tau \mathrm{e}^{-\lambda s}\, l(y_x^\alpha(s), \alpha(s)) ds + \mathrm{e}^{-\lambda \tau} v(t-\tau, y_x^\alpha(\tau)). \end{aligned}$$

Since the above inequality holds for any $\alpha \in \mathcal{A}$, taking the infimum among $\alpha \in \mathcal{A}$, we get

$$v(t,x) \geq \inf_{\alpha \in \mathcal{A}} \left\{ \int_0^\tau \mathrm{e}^{-\lambda s} l(y_x^\alpha(s), \alpha(s)) ds + \mathrm{e}^{-\lambda \tau} v(t-\tau, y_x^\alpha(\tau)) \right\}.$$

To prove the opposite inequality, we set $y := y_x^\alpha(\tau)$, then for any $\varepsilon > 0$ there exists $\overline{\alpha}_\varepsilon \in \mathcal{A}$ such that

$$v(t-\tau, y) + \varepsilon \geq J(t-\tau, y; \overline{\alpha}_\varepsilon).$$

We define a new control on $[0,+\infty)$ (still depending on ε) as

$$\overline{\alpha} := \begin{cases} \alpha(s), & 0 \le s \le \tau, \\ \overline{\alpha}_\varepsilon(s-\tau), & s > \tau. \end{cases}$$

Recall that, by the semigroup property, $y_x^{\overline{\alpha}}(t) = y_y^{\overline{\alpha}_\varepsilon}(t-\tau)$.
Then, for any $\varepsilon > 0$,

$$\begin{aligned} v(t,x) &\le J(t,x;\overline{\alpha}) \\ &= \int_0^\tau \mathrm{e}^{-\lambda s} l(y_x^\alpha(s),\alpha(s))ds + \int_\tau^t \mathrm{e}^{-\lambda s} l(y_y^{\overline{\alpha}_\varepsilon}(s),\overline{\alpha}_\varepsilon(s))ds + e^{-\lambda t} g(y_y^{\overline{\alpha}_\varepsilon}(t-\tau)) \\ &= \int_0^\tau \mathrm{e}^{-\lambda s} l(y_x^\alpha(s),\alpha(s))ds + e^{-\lambda\tau} J(t-\tau,y;\overline{\alpha}_\varepsilon) \\ &\le \int_0^\tau \mathrm{e}^{-\lambda s} l(y_x^\alpha(s),\alpha(s))ds + e^{-\lambda\tau} v(t-\tau,y) + e^{-\lambda\tau}\epsilon. \end{aligned}$$

Passing to the limit as $\varepsilon \to 0^+$ and then taking the infimum among $\alpha \in \mathcal{A}$, we can deduce

$$v(t,x) \le \inf_{\alpha\in\mathcal{A}} \left\{ \int_0^\tau \mathrm{e}^{-\lambda s} l(y_x^\alpha(s),\alpha(s))ds + \mathrm{e}^{-\lambda\tau} v(t-\tau, y_x^\alpha(\tau)) \right\},$$

and so we can conclude (20). □

Next we are going to characterize the value function $v(t,x)$ as the unique viscosity solution of the Hamilton-Jacobi-Bellman equation:

$$u_t + \lambda u + H(x,Du) = 0, \tag{21}$$

where the Hamiltonian is given by

$$H(x,p) = \sup_{a\in A} \big\{ -f(x,a)\cdot p - l(x,a) \big\}. \tag{22}$$

Theorem 3.1. Under the same assumptions of Proposition 3.1, then the value function (19) is a viscosity solution of the equation (21), satisfying the initial condition $v(0,x) = g(x)$, (see Remark 3.4).

Proof. The idea is to show both the subsolution and the supersolution properties, using the Dynamic Programming Principle.

I. We prove that $v(t,x)$ is a *viscosity subsolution:* Let $\varphi \in C^1(\mathbb{R}^n \times (0,+\infty))$ be such that $v - \varphi$ has a local maximum at some point (t_0,x_0), i.e. there exists $R > 0$ such that

$$v(t,x) - \varphi(t,x) \le v(t_0,x_0) - \varphi(t_0,x_0), \quad \forall\, |x-x_0| < R, |t-t_0| < R,$$

which is equivalent to say:

$$v(t_0, x_0) - v(t, x) \geq \varphi(t_0, x_0) - \varphi(t, x), \quad \forall\, |x - x_0| < R, |t - t_0| < R. \quad (23)$$

Now, for any $a \in A$, let us consider the trajectories corresponding to the constant control $\alpha(t) = a$, i.e. $y^a_{x_0}$ solving:

$$\begin{cases} \dot{y}(t) = f(y(t), a), & t > 0, \\ y(0) = x_0. \end{cases}$$

Fix $0 \leq \tau \leq t_0$ small enough that $|t_0 - (t_0 - \tau)| = \tau < R$ and $|y^a_{x_0}(\tau) - x_0| < R$. Then the inequality (23) holds with $x = y^a_{x_0}(\tau)$ and $t = t_0 - \tau$. Using (20) to estimate $v(t_0, x_0)$, we can deduce

$$\begin{aligned} \varphi(t_0, x_0) - \varphi(t_0 - \tau, y^a_{x_0}(\tau)) &\leq v(t_0, x_0) - v(t_0 - \tau, y^a_{x_0}(\tau)) \\ &\leq \int_0^\tau \mathrm{e}^{-\lambda s} l(y^a_{x_0}(s), a) ds + \big(\mathrm{e}^{-\lambda \tau} - 1\big) v(t_0 - \tau, y^a_{x_0}(\tau)). \end{aligned}$$

Now we add and subtract $\varphi(t_0 - \tau, x - 0)$ (in such a way we vary one variable at a time) and we divide by $\tau > 0$, which means

$$\begin{aligned} &\frac{\varphi(t_0, x_0) - \varphi(t_0 - \tau, x_0)}{\tau} + \frac{\varphi(t_0 - \tau, x_0) - \varphi(t_0 - \tau, y^a_{x_0}(\tau))}{\tau} \\ &\qquad \leq \frac{1}{\tau} \int_0^\tau \mathrm{e}^{-\lambda s} l(y^a_{x_0}(s), a) ds + \frac{\mathrm{e}^{-\lambda \tau} - 1}{\tau} v(t_0 - \tau, y^a_{x_0}(\tau)). \end{aligned}$$

Passing to the limit as $\tau \to 0^+$ and by using that $\varphi \in C^1$, the continuity of the functions $y^a_{x_0}(\cdot)$, $l(y^a_{x_0}(\cdot), a)$ and $v(\cdot, x)$ and the fact that $\dot{y}^a_{x_0}(\tau) = f(y^\alpha_x(\tau), a)$ is continuous too (since the control is constant), then we can conclude

$$\varphi_t(t_0, x_0) + \lambda v(t_0, x_0) - D\varphi(t_0, x_0) \cdot f(x_0, a) - l(x_0, a) \leq 0.$$

Since the above inequality is true for any $a \in A$, taking the supremum in A, we get

$$\varphi_t(t_0, x_0) + \lambda v(t_0, x_0) + H(x_0, D\varphi(t_0, x_0)) \leq 0;$$

therefore $v(t, x)$ is a viscosity subsolution of the equation (21).

II. We prove that $v(t, x)$ is a *viscosity supersolution:* Let $\varphi \in C^1(\mathbb{R}^n \times (0, +\infty))$ be such that $v - \varphi$ has a local minimum at some point (t_0, x_0), i.e. there exists $R > 0$ such that

$$v(t, x) - \varphi(t, x) \geq v(t_0, x_0) - \varphi(t_0, x_0), \quad \forall\, |x - x_0| < R, |t - t_0| < R,$$

which is equivalent to say:

$$v(t_0,x_0)-v(t,x)\le \varphi(t_0,x_0)-\varphi(t,x),\quad \forall\,|x-x_0|<R, |t-t_0|<R. \tag{24}$$

We want to use the Dynamic Programming Principle to estimate $v(t_0,x_0)$ from below. Therefore, by the definition of infimum, for any $\varepsilon>0$ and $\tau>0$, there exists a control $\overline{\alpha}(t)=\alpha_{\varepsilon,\tau}(t)$ such that

$$v(t_0,x_0)\ge \int_0^\tau \mathrm{e}^{-\lambda s} l(y_{x_0}^{\overline{\alpha}}(s),\overline{\alpha}(s))ds+\mathrm{e}^{-\lambda\tau}v(t_0-\tau,y_{x_0}^{\overline{\alpha}}(\tau))-\tau\varepsilon.$$

Choosing $\tau>0$ small enough that (24) holds, we can write

$$\begin{aligned}&\big(\varphi(t_0,x_0)-\varphi(t_0-\tau,x_0)\big)+\big(\varphi(t_0-\tau,x_0)-\varphi(t_0-\tau,y_{x_0}^{\overline{\alpha}}(\tau))\big)+\\ &-\int_0^\tau \mathrm{e}^{-\lambda s} l(y_{x_0}^{\overline{\alpha}}(s),\overline{\alpha}(s))ds+\big(1-\mathrm{e}^{-\lambda\tau}\big)v(t_0-\tau,y_{x_0}^{\overline{\alpha}}(\tau))+\tau\varepsilon\ge 0.\end{aligned} \tag{25}$$

To conclude is a bit more technical than in the case of the subsolution-property because this time the control is not constant and it depends on ε and τ. We need to use the Lipschitz continuity of $l(x,a)$ and $f(x,a)$ and the properties (iii) and (iv) for the trajectories $y_x^\alpha(t)$, which tell that $y_x^\alpha(t)$ is locally bounded, uniformly w.r.t. the control and locally uniformly w.r.t. x (property (iii)) and that

$$|y_{x_0}^{\alpha}(\tau)-x_0|\le M_{x_0}\,\tau,$$

for any $\alpha\in\mathcal{A}$ and for $\tau>0$ small enough and where M_{x_0} is a positive constant depending just on f and x_0 (property (iv)).
Let us first look at the integral-term:

$$\begin{aligned}&-\int_0^\tau \mathrm{e}^{-\lambda s} l(y_{x_0}^{\overline{\alpha}}(s),\overline{\alpha}(s))ds\le -\int_0^\tau \mathrm{e}^{-\lambda s} l(x_0,\overline{\alpha}(s))ds+\\ &+\int_0^\tau \mathrm{e}^{-\lambda s}|l(y_{x_0}^{\overline{\alpha}}(s),\overline{\alpha}(s))-l(x_0,\overline{\alpha}(s))|ds\le -\int_0^\tau \mathrm{e}^{-\lambda s} l(x_0,\overline{\alpha}(s))ds+\\ &+\mathrm{Lip}(l)M_{x_0}\int_0^\tau s\,ds\le -\int_0^\tau \mathrm{e}^{-\lambda s} l(x_0,\overline{\alpha}(s))ds+\mathrm{Lip}(l)M_{x_0}\frac{\tau^2}{2}\\ &=-\int_0^\tau l(x_0,\overline{\alpha}(s))ds+\int_0^\tau \big(1-\mathrm{e}^{-\lambda s}\big)l(x_0,\overline{\alpha}(s))ds+\mathrm{Lip}(l)M_{x_0}\frac{\tau^2}{2}\\ &\le -\int_0^\tau l(x_0,\overline{\alpha}(s))ds+C_l\int_0^\tau \lambda s ds+\mathrm{Lip}(l)M_{x_0}\frac{\tau^2}{2}\\ &=-\int_0^\tau l(x_0,\overline{\alpha}(s))ds+\big(C_l\lambda+\mathrm{Lip}(l)M_{x_0}\big)\frac{\tau^2}{2}\end{aligned} \tag{26}$$

where we have used that $l(x,a)$ is bonded uniformly w.r.t. a and that $1 - \mathrm{e}^{-\lambda s} \leq \lambda s$.
For sake of simplicity, let us call $C_1 := \big(C_l \lambda + \mathrm{Lip}(l) M_{x_0}\big)/2$.
Now we need to estimate:

$$\begin{aligned}
\varphi(t_0-\tau,x_0) - \varphi(t_0-\tau, y^{\overline{\alpha}}_{x_0}(\tau)) &= -\int_0^\tau D\varphi(y^{\overline{\alpha}}_{x_0}(s)) \cdot f(y^{\overline{\alpha}}_{x_0}(s),\overline{\alpha}(s))ds \\
= -\int_0^\tau D\varphi(x_0)\cdot f(x_0,\overline{\alpha}(s))ds &+ \int_0^\tau \big[D\varphi(x_0) - D\varphi(y^{\overline{\alpha}}_{x_0}(s))\big]\cdot f(x_0,\overline{\alpha}(s))ds + \\
&+ \int_0^\tau D\varphi(y^{\overline{\alpha}}_{x_0}(s)) \cdot \big[f(x_0,\overline{\alpha}(s)) - f(y^{\overline{\alpha}}_{x_0}(s),\overline{\alpha}(s))\big]ds \\
\leq -\int_0^\tau D\varphi(x_0)\cdot f(x_0,\overline{\alpha}(s))ds &+ C_{1,x_0} M_{x_0}\frac{\tau^2}{2} + \mathrm{Lip}(f) M_{x_0} C_{2,x_0}\frac{\tau^2}{2} \\
&= -\int_0^\tau D\varphi(x_0)\cdot f(x_0,\overline{\alpha}(s))ds + C_2\tau^2, \quad (27)
\end{aligned}$$

where C_{1,x_0} depends on a local bound for f and a local bound for $D\varphi$ (assuming $\varphi \in C^2$ and the local maximum strict) while $C_{2,x_0} := \max\{|D\varphi(x)| \mid |x - x_0| \leq C\}$ and $C > 0$ is a constant independent of the control and τ, which is given by the property (iii) for the trajectories $y^{\alpha}_x(t)$.
Since $\alpha(s) \in A$, for any $s > 0$, we can estimate the sum of (26) and (27) by taking the supremum for $a \in A$, which gives:

$$\begin{aligned}
\varphi(t_0-\tau,x_0) - \varphi(t_0-\tau, y^{\overline{\alpha}}_{x_0}(\tau)) - \int_0^\tau \mathrm{e}^{-\lambda s} l(y^{\overline{\alpha}}_{x_0}(s),\overline{\alpha}(s))ds \\
\leq H(x_0, D\varphi(x_0))\,\tau + \big(C_1 + C_2\big)\tau^2
\end{aligned} \quad (28)$$

Using (28) in (25) and dividing by $\tau > 0$, we find

$$\begin{aligned}
\frac{\varphi(t_0,x_0) - \varphi(t_0-\tau,x_0)}{\tau} - \frac{\mathrm{e}^{-\lambda\tau} - 1}{\tau} v(t_0-\tau, y^{\overline{\alpha}}_{x_0}(\tau)) + H(x_0, D\varphi(x_0)) \\
+ \big(C_1 + C_2\big)\tau + \varepsilon \geq 0,
\end{aligned}$$

which gives the supersolution-property passing to the limit as $\tau \to 0^+$ and then as $\varepsilon \to 0^+$. □

We rest prove that the value function is the unique viscosity solution of the Hamilton-Jacobi-Bellman equation (21) having the terminal cost as initial condition. In such a way, one can solve the corresponding Hamilton-Jacobi-Bellman equation in order to find the value function of the associated control problem.

Theorem 3.2 (Comparison Principles)**.** Let $H : \mathbb{R}^{2n} \to \mathbb{R}$ continuous and satisfying (22) with $f(x,a)$ and $l(x,a)$ continuous, Lipschitz in x uniformly w.r.t. $a \in A$ and $l(x,a)$ bounded in x uniformly w.r.t. $a \in A$ and $\lambda \geq 0$.
Let $u_1, u_2 : (0,+\infty) \times \mathbb{R}^n \to \mathbb{R}$ be continuous and bounded in $[0,T] \times \mathbb{R}^n$ for any $T > 0$.
If u_1 and u_2 are, respectively, a viscosity subsolution and a viscosity supersolution of the equation (21) on $[0,+\infty) \times \mathbb{R}^n$, then, for any $T > 0$,

$$\sup_{[0,T]\times\mathbb{R}^n} (u_1 - u_2) \leq \sup_{\{0\}\times\mathbb{R}^n} (u_1 - u_2)^+. \tag{29}$$

Comparison Principles implies uniqueness for the viscosity solutions of the corresponding Cauchy problem.

Corollary 3.1. *Under the assumptions of Theorem 3.2 and given* $g : \mathbb{R}^n \to \mathbb{R}$ *continuous and bounded, there exists a unique continuous and bounded viscosity solution of the equation* (21) *on* $[0,T] \times \mathbb{R}^n$*, such that* $u(0,x) = g(x)$*.*

Proof. It is sufficient to apply Theorem 3.2 to two different viscosity solutions u_1 and u_2 with the same initial condition, i.e. such that

$$\sup_{\{0\}\times\mathbb{R}^n} |u_1(x) - u_2(x)| = \sup_{\mathbb{R}^n} |g(x) - g(x)| = 0.$$

$\square$

Proof of Theorem 3.2 (Comparison Principles). We prove the theorem by contradiction: this means that we are going to assume (29) false and then to build two particular test-functions (one for u_1 and the other one for u_2) which will lead to a contradiction.

Let us start studying the following continuous function:

$$\Phi(t,s,x,y) := u_1(t,x) - u_2(s,y) + \\ - \frac{|t-s|^2 + |x-y|^2}{2\varepsilon} - \beta\big((1+|x|^2)^{\frac{m}{2}} + (1+|y|^2)^{\frac{m}{2}}\big) - \eta(t+s),$$

where ε, β, m, η positive parameters to choose later in a suitable way.
Note that the above function is a function defined on $[0,+\infty) \times [0,+\infty) \times \mathbb{R}^n \times \mathbb{R}^n$. This method is called "doubling of variables" and it is typical for proving comparison principles for viscosity solutions.

Note also that

$$\Phi(t,s,x,y) \leq u_1(t,x) - u_2(s,y), \quad \forall\,(s,t,x,y) \in [0,+\infty)^2 \times \mathbb{R}^{2n}. \tag{30}$$

We define

$$A := \sup_{\{0\}\times\mathbb{R}^n} (u_1 - u_2)^+ \geq 0.$$

and we assume for contradiction that (29) does not hold, which means that there exists $\delta > 0$ and $\widetilde{t} \in (0, T]$ and $\widetilde{x} \in \mathbb{R}^n$ such that

$$u_1(\widetilde{t}, \widetilde{x}) - u_2(\widetilde{t}, \widetilde{x}) = A + \delta.$$

Now let us choose $\beta > 0$ and $\eta > 0$ such that $2\beta(1 + |\widetilde{x}|^2)^{m/2} + 2\eta\widetilde{t} \leq \delta/2$, for all $m \leq 1$, which implies

$$\Phi(\widetilde{t}, \widetilde{t}, \widetilde{x}, \widetilde{x}) \geq A + \delta - \frac{\delta}{2} = A + \frac{\delta}{2}; \tag{31}$$

therefore

$$\sup_{[0,T]^2\times\mathbb{R}^{2n}} \Phi(t, s, x, y) \geq A + \frac{\delta}{2}. \tag{32}$$

Moreover, $\Phi(t, s, x, y) \to -\infty$, whenever $|x| + |y| \to +\infty$, hence there exists $(\overline{t}, \overline{s}, \overline{x}, \overline{y})$ such that

$$\Phi(\overline{t}, \overline{s}, \overline{x}, \overline{y}) = \sup_{[0,T]^2\times\mathbb{R}^{2n}} \Phi(t, s, x, y) \geq A + \frac{\delta}{2} > 0.$$

Note that $u_1(t, x) - u_2(s, y) - \beta\left((1 + |x|^2)^{\frac{m}{2}} + (1 + |y|^2)^{\frac{m}{2}}\right) \geq \Phi(t, s, x, y)$, for any s, t, x, y, then, writing the previous inequality in $(\overline{t}, \overline{s}, \overline{x}, \overline{y})$, we find

$$\beta\left((1 + |\overline{x}|^2)^{\frac{m}{2}} + (1 + |\overline{y}|^2)^{\frac{m}{2}}\right) \leq \sup_{[0,T]\times\mathbb{R}^n} u_1 - \inf_{[0,T]\times\mathbb{R}^n} u_2 - A := C_1, \tag{33}$$

for any $\varepsilon > 0$ and $m \in (0, 1]$.
Moreover by (30) and (32), we can observe that $C_1 > 0$.

¿From (33) we find $\overline{x}, \overline{y} \in \overline{B} := \overline{B}(0, (C_1/\beta)^{1/m})$, in fact

$$|\overline{x}|^m \leq (1 + |\overline{x}|^2)^{\frac{m}{2}} \leq (1 + |\overline{x}|^2)^{\frac{m}{2}} + (1 + |\overline{y}|^2)^{\frac{m}{2}} \leq \frac{C_1}{\beta}, \tag{34}$$

and analogously for $\overline{y}$.

Now we want to show that the maximum point $(\overline{t}, \overline{s}, \overline{x}, \overline{y})$ is near the diagonal, in a suitable sense, w.r.t. $\varepsilon > 0$.
First we remark that, since u_1 and u_2 are continuous in a compact set, then

they are both uniformly continuous, therefore there exists a modulus of continuity $\omega(\cdot)$ (that we can assume to be the same, up to consider the sum of the two moduli of continuity which is still a modulus of continuity) such that

$$u_i(t,x)-u_i(s,y)=\omega(|t-s|+|x-y|), \quad \forall t,s\in[0,T], x,y\in\overline{B}, \text{ and for } i=1,2. \tag{35}$$

Since $(\overline{t},\overline{s},\overline{x},\overline{y})$ is the maximum point for Φ, it is trivial that

$$\Phi(\overline{t},\overline{t},\overline{x},\overline{x})-\Phi(\overline{s},\overline{s},\overline{y},\overline{y})\leq 2\Phi(\overline{t},\overline{s},\overline{x},\overline{y}),$$

which (writing down explicitly the functions in the above inequality) means

$$\begin{aligned}\frac{|\overline{x}-\overline{y}|^2+|\overline{t}-\overline{s}|^2}{\varepsilon} &\leq u_1(\overline{t},\overline{x})-u_1(\overline{s},\overline{y})+u_2(\overline{t},\overline{x})-u_2(\overline{s},\overline{y})\\ &\leq 2\sup_{[0,T]\times\mathbb{R}^n}u_1+2\sup_{[0,T]\times\mathbb{R}^n}u_2=\frac{C^2}{\sqrt{2}}, \end{aligned}\tag{36}$$

since u_1 and u_2 are continuous and so bounded in the compact set $[0,T]\times\overline{B}$.

Now we are going to use (35) in order to improve (36). In fact, using the concavity of $\sqrt{\ }$, (36) implies:

$$|\overline{x}-\overline{y}|+|\overline{t}-\overline{s}|\leq C\sqrt{\varepsilon}. \tag{37}$$

Therefore, by (35) and (37), then the estimate (36) becomes:

$$\frac{|\overline{x}-\overline{y}|^2+|\overline{t}-\overline{s}|^2}{\varepsilon}\leq 2\,\omega(C\sqrt{\varepsilon}). \tag{38}$$

Now the idea is to use that u_1 and u_2 are, respectively, a viscosity subsolution and a viscosity supersolution of the equation (21), and to build suitable test-functions at the points $(\overline{t},\overline{x})$ and $(\overline{s},\overline{y})$, respectively.

So the first step is to show that $\overline{t}\neq 0$ and $\overline{s}\neq 0$. This is easy to prove using the above estimates (in particular (35) and (37)) and (31) (that we have assumed for contradiction of (29)). In fact, let us assume for contradiction that $\overline{t}=0$, then

$$\begin{aligned}&\Phi(0,\overline{s},\overline{x},\overline{y})\leq u_1(0,\overline{x})-u_2(\overline{s},\overline{y})=u_1(0,\overline{x})-u_2(0,\overline{x})+u_2(0,\overline{x})-u_2(\overline{s},\overline{y})\\ &\leq\sup_{\{0\}\times\mathbb{R}^n}(u_1-u_2)+\big(u_2(0,\overline{x})-u_2(\overline{s},\overline{y})\big)\leq A+\omega(|\overline{s}|+|\overline{x}-\overline{y}|)\leq A+\omega(C\sqrt{\varepsilon}).\end{aligned}$$

Therefore, choosing $\varepsilon > 0$ small enough that $\omega(C\sqrt{\varepsilon}) \leq \frac{\delta}{4} < \frac{\delta}{2}$, then the above inequality gives

$$\Phi(\overline{t}, \overline{s}, \overline{x}, \overline{y}) = \Phi(0, \overline{s}, \overline{x}, \overline{y}) < A + \frac{\delta}{2},$$

which contradicts (31). To prove $\overline{s} \neq 0$ is exactly the same.

Next we define the following test-functions:

$$\varphi(t,x) := u_2(\overline{s}, \overline{y}) + \frac{|x-\overline{y}|^2 + |t-\overline{s}|^2}{2\varepsilon} + \beta\big((1+|x|^2)^{\frac{m}{2}} + (1+|\overline{y}|^2)^{\frac{m}{2}}\big) + \eta(t+\overline{s}),$$
$$\psi(s,y) := u_1(\overline{t}, \overline{x}) - \frac{|\overline{x}-y|^2 + |\overline{t}-s|^2}{2\varepsilon} - \beta\big((1+|\overline{x}|^2)^{\frac{m}{2}} - (1+|y|^2)^{\frac{m}{2}}\big) - \eta(\overline{t}+s).$$

It is not difficult to show that $(u_1 - \varphi)(t,x)$ attends maximum at $(\overline{t}, \overline{x})$ while $(u_2 - \psi)(s,y)$ attends minimum at $(\overline{s}, \overline{y})$. Therefore, using that u_1 is a viscosity subsolution and u_2 is a viscosity supersolution, we get:

$$\varphi_t(\overline{t}, \overline{x}) + \lambda u_1(\overline{t}, \overline{x}) + H(\overline{x}, D\varphi(\overline{t}, \overline{x})) \leq 0,$$
$$\psi_t(\overline{s}, \overline{y}) + \lambda u_2(\overline{s}, \overline{y}) + H(\overline{y}, D\psi(\overline{s}, \overline{y})) \geq 0;$$

subtracting the second one to the first one, we can write

$$0 \geq \varphi_t(\overline{t}, \overline{x}) - \psi_t(\overline{s}, \overline{y}) + \lambda\big(u_1(\overline{t}, \overline{x}) - u_2(\overline{s}, \overline{y})\big) + H(\overline{x}, D\varphi(\overline{t}, \overline{x})) - H(\overline{y}, D\psi(\overline{s}, \overline{y})). \tag{39}$$

Now we rest to calculate the derivatives of the test-functions and to estimate in a suitable way the difference of the two Hamiltonian-terms using the structure of H. So first note

$$\varphi_t(\overline{t}, \overline{x}) = \frac{\overline{t} - \overline{s}}{\varepsilon} + \eta,$$
$$\psi_t(\overline{s}, \overline{y}) = \frac{\overline{t} - \overline{s}}{\varepsilon} - \eta,$$

and

$$D\varphi(\overline{t}, \overline{x}) = \frac{\overline{x} - \overline{y}}{\varepsilon} + m\beta(1+|\overline{x}|^2)^{\frac{m-2}{2}}\overline{x} = \frac{\overline{x} - \overline{y}}{\varepsilon} + \gamma\overline{x},$$
$$D\psi(\overline{s}, \overline{y}) = \frac{\overline{x} - \overline{y}}{\varepsilon} - m\beta(1+|\overline{y}|^2)^{\frac{m-2}{2}}\overline{y} = \frac{\overline{x} - \overline{y}}{\varepsilon} - \tau\overline{y},$$

where, for sake of simplicity, we have written $\gamma = m\beta(1+|\overline{x}|^2)^{\frac{m-2}{2}}$ and $\tau = m\beta(1+|\overline{y}|^2)^{\frac{m-2}{2}}$.

Then the inequality (39) becomes:

$$
\begin{aligned}
0 &\geq 2\eta + \lambda\big(u_1(\bar{t},\overline{x}) - u_2(\overline{s},\overline{y})\big) + H\left(\overline{x}, \frac{\overline{x}-\overline{y}}{\varepsilon} + \gamma\overline{x}\right) - H\left(\overline{y}, \frac{\overline{x}-\overline{y}}{\varepsilon} - \tau\overline{y}\right) \\
&\geq 2\eta + \lambda\bigg(A + \frac{\delta}{2}\bigg) + H\left(\overline{x}, \frac{\overline{x}-\overline{y}}{\varepsilon} + \gamma\overline{x}\right) - H\left(\overline{y}, \frac{\overline{x}-\overline{y}}{\varepsilon} - \tau\overline{y}\right), \quad (40)
\end{aligned}
$$

using (30) and (32).

To conclude we use the following property for the Hamiltonian $H(x,p)$ (that we will verify at the end of the proof):

$$
\begin{aligned}
&H\left(y, \frac{x-y}{\varepsilon} - \tau y\right) - H\left(x, \frac{x-y}{\varepsilon} + \gamma x\right) \\
&\quad \leq \frac{1}{\varepsilon}\mathrm{Lip}(f)|x-y|^2 + \mathrm{Lip}(l)|x-y| + \gamma K(1+|x|^2) + \tau K(1+|y|^2). \quad (41)
\end{aligned}
$$

Using (41) in (40) we can conclude; in fact

$$
2\eta \leq \frac{1}{\varepsilon}\mathrm{Lip}(f)|\overline{x}-\overline{y}|^2 + \mathrm{Lip}(l)|\overline{x}-\overline{y}| + \gamma K(1+|\overline{x}|^2) + \tau K(1+|\overline{y}|^2), \quad (42)
$$

where we have used that $A + \frac{\delta}{2} \geq 0$.

We can estimate the first two terms in the above inequality by (38) and (37), respectively; then, using the definition of γ and τ, the inequality (42) becomes

$$
2\eta \leq 2\mathrm{Lip}(f)\,\omega(C\sqrt{\varepsilon}) + \mathrm{Lip}(l)C\sqrt{\varepsilon} + m\beta K\left((1+|\overline{x}|^2)^{\frac{m}{2}} + (1+|\overline{y}|^2)^{\frac{m}{2}}\right). \quad (43)
$$

Now we use (34) in (43) getting

$$
2\eta \leq 2\mathrm{Lip}(f)\,\omega(C\sqrt{\varepsilon}) + \mathrm{Lip}(l)C\sqrt{\varepsilon} + 2m\beta K\frac{C_1}{\beta} = \widetilde{\omega}(\varepsilon) + 2mKC_1, \quad (44)
$$

where $\widetilde{\omega}(\varepsilon) \to 0$, as $\varepsilon \to 0^+$.

Then we can choose $m \leq \min\{\eta/(2KC_1), 1\}$, and by (44) we can conclude

$$
2\eta \leq \widetilde{\omega}(\varepsilon) + \eta
$$

which, passing to the limit as $\varepsilon \to 0^+$, contradicts the fact that $\eta > 0$.

Now we remain to prove that if $H(x,p)$ is given by (22), then the property (41) is satisfied.
The claim is easy to prove using the Lipschitz continuity of $f(x,a)$ and $l(x,a)$ w.r.t. x. Note that $H(x,p)$ is given as supremum, then, for any $\overline{\varepsilon} > 0$, there exists $\overline{a} = \overline{a}_\varepsilon \in A$ control such that

$$H\left(y, \frac{x-y}{\varepsilon} + \tau y\right) \le -f(y,\overline{a}) \cdot \left(\frac{x-y}{\varepsilon} + \tau y\right) - l(y,\overline{a}) + \overline{\varepsilon},$$

hence

$$\begin{aligned}
H&\left(y, \frac{x-y}{\varepsilon} + \tau y\right) - H\left(x, \frac{x-y}{\varepsilon} + \gamma x\right) \\
&\le -f(y,\overline{a}) \cdot \left(\frac{x-y}{\varepsilon} + \tau y\right) + f(x,\overline{a}) \cdot \left(\frac{x-y}{\varepsilon} + \gamma x\right) - l(y,\overline{a}) + l(x,\overline{a}) + \overline{\varepsilon} \\
&\le \frac{1}{\varepsilon}\mathrm{Lip}(f)|x-y|^2 + \mathrm{Lip}(l)|x-y| - \tau f(y,\overline{a}) \cdot y + \gamma f(x,\overline{a}) \cdot x + \overline{\varepsilon} \\
&\le \frac{1}{\varepsilon}\mathrm{Lip}(f)|x-y|^2 + \mathrm{Lip}(l)|x-y| + \tau K(1+|y|^2) + \gamma K(1+|x|^2) + \overline{\varepsilon},
\end{aligned}$$

using that $f(z,a) \cdot z \le \mathrm{Lip}(f)|z|^2 + \sup_{a\in A}|f(0,a)|\,|z| \le K(1+|z|^2)$ (note $|z| \le (1+|z|^2)$ always) and passing to the limit as $\overline{\varepsilon} \to 0^+$.
Then claim (41) is proved and so the theorem is. □

For more information on optimal control problems and Hamilton-Jacobi-Bellman equations, we refer to the book of Bardi and Capuzzo Dolcetta, Chapter 2 I and III. For sake of completeness, we in particular suggest to see Section I.6 for a necessary and sufficient condition for the existence of optimal trajectories (namely the Pontryagin Maximum Principle).

4 The Hopf-Lax formula.

In this section we are going to study the Cauchy problem for first-order PDEs, in the particular case when the Hamiltonian depends only on the gradient, i.e.

$$H(x,z,p) = H(p).$$

This means we want to study the viscosity solution of the Cauchy problem:

$$\begin{cases} u_t + H(Du) = 0, & x \in \mathbb{R}^n,\ t > 0, \\ u(0,x) = g(x), & x \in \mathbb{R}^n. \end{cases} \tag{45}$$

Let us assume that the Hamiltonian H is continuous and satisfies

$$p \mapsto H(p) \quad \text{convex} \qquad \text{and} \quad \lim_{|p|\to+\infty} \frac{H(p)}{|p|} = +\infty. \tag{46}$$

Example 4.1. The main example to bear in mind is

$$H(p) = \frac{1}{\alpha}|p|^{\alpha}, \quad \text{with } \alpha > 1. \tag{47}$$

Example 4.2. Another family of Hamiltonians satisfying (46) is:

$$H(p) = \mathrm{e}^{\lambda\,|p|} + C,$$

for any $\lambda > 0$ and $C \in \mathbb{R}$.

Next we recall the definition of the Legendre-Fenchel transform, which one of the main notion in convex analysis.

Definition 4.1 (Legendre-Fenchel transform)**.**
Let us assume that $H : \mathbb{R}^n \to \mathbb{R}$ is continuous, then we call Legendre-Fenchel transform the following function:

$$H^*(q) = \sup_{q\in\mathbb{R}^n} [q \cdot p - H(p)]. \tag{48}$$

Example 4.3. The Legendre-Fenchel transform of the Hamiltonian given by formula (47) is

$$H^*(q) = \frac{1}{\beta}|q|^{\beta}, \tag{49}$$

where $\beta = \frac{\alpha}{\alpha-1} > 1$.
In particular, whenever $\alpha = 2$, then $\beta = 2$: therefore in this case $H = H^*$.

Proposition 4.1 (Properties of the Legendre-Fenchel transform)**.** Under assumptions (46), the same two properties are true for H^*, that means

$$q \mapsto H^*(q) \quad \text{convex} \qquad \text{and} \quad \lim_{|q|\to+\infty} \frac{H^*(q)}{|q|} = +\infty. \tag{50}$$

Moreover $H^{**} = H$; we say that the Legendre-Fenchel transform is *involutive.*

For a proof of the previous results, one can see the book of Evans (Theorem 3, Section 3.3).

Using the above remark and setting $L := H^*$, we can write

$$H(p) = L^*(p) = \sup_{q\in\mathbb{R}^n} \big\{p\cdot q - L(q)\big\}; \tag{51}$$

in this way any Hamiltonian depending just on the gradient can be written as a Hamiltonian of an Hamilton-Jacobi-Bellman equation and therefore associated to a control problem, with $f(x,a) = -a$, $l(x,a) = L(a) = H^*(a)$ and where $a = q \in \mathbb{R}^n$. Then the set where the controls take values is a priori not compact. Nevertheless, by the following remark we can partially overcome such a problem.

Remark 4.1. By assumption (46) it is easy to show that the supremum in (51) is attained in some compact $K \subset \mathbb{R}^n$, i.e. there exists $R >> 0$ such that

$$H(p) = \sup_{q\in\mathbb{R}^n} \big\{p\cdot q - L(q)\big\} = \sup_{q\in\overline{B_R}(p)} \big\{p\cdot q - L(q)\big\}. \tag{52}$$

Note that in general the compact set where the supremum is attained depends on p. If one can show that $|Du| \le C$, then p is in a compact set and the supremum in (51) is attained in a compact set independent of p.

Let now write the associated control system and value function, which are, respectively,

$$\begin{cases} \dot y(t) = -\alpha(t), & t>0 \\ y(0) = x \end{cases} \quad\Longrightarrow\quad y_x^\alpha(t) = x - \int_0^t \alpha(s)ds;$$

and

$$v(t,x) = \inf_{\alpha\in\mathcal{A}} \left\{ \int_0^t L(\alpha(s))ds + g\left(x - \int_0^t \alpha(s)ds\right)\right\}. \tag{53}$$

Now we are going to manipulate the expression in (53), to write this in a more "calculus of variation form".

First we set $y := x - \int_0^t \alpha(s)ds$ and define $\xi(s) := x - \int_0^s \alpha(s')ds$;: note that $\xi(0) = x$, $\xi(t) = y$ and $\dot\xi(t) = -\alpha(t)$, for any $t > 0$.
Let us denote by $AC([0,+\infty);\mathbb{R}^n)$ the set of all the functions $\xi : [0,+\infty) \to \mathbb{R}^n$ absolutely continuous.
Therefore, (53) can be written as

$$v(t,x) = \inf\left\{ \int_0^t L\big(-\dot\xi(s)\big)ds + g\,(y) \;\big|\, \xi\in AC([0,+\infty);\mathbb{R}^n)\ \xi(0) = x,\ \xi(t) = y \right\}. \tag{54}$$

Fix $t > 0$ and define $w(s) := \xi(t-s)$, this new curve is such that

$$\dot{w}(s) = -\dot{\xi}(s) = \alpha(s), \quad w(0) = y, \quad w(t) = x.$$

Hence the infimum in (54) can be equivalently written as

$$v(t,x) = \inf\left\{ \int_0^t L\big(\dot{w}(s)\big)ds + g\,(y) \;\big|\; w \in AC([0,+\infty);\mathbb{R}^n)\; w(0) = y,\; w(t) = x \right\}. \tag{55}$$

Note that, in the above formulation g is not anymore considered as "terminal cost" but as "initial cost".

In the next theorem, we show that the infimum given by (55) (which is an infimum in an infinite-dimensional space) can indeed be expressed by an infimum in $\mathbb{R}^n$.

Theorem 4.1. Assuming H is continuous and convex, for any $t > 0$ and any $x \in \mathbb{R}^n$, if $v(t,x)$ is the value function given in (55), then

$$v(t,x) = \inf_{y\in\mathbb{R}^n}\left[g(y) + tH^*\left(\frac{x-y}{t}\right)\right] =: u(t,x). \tag{56}$$

We call the infimum on the right-hand side *Hopf-Lax function.*

Proof. Recall that, by definition $L(q) = H^*(q)$.

We first show that $v(t,x) \leq u(t,x)$.
Fix $y \in \mathbb{R}^n$ and look at $\overline{w}(s) := y + s(x-y)/t$ with $0 \leq s \leq t$:
note that $\overline{w}$ is absolutely continuous and $\overline{w}(0) = y$ and $\overline{w}(t) = x$, hence $\overline{w}$ is a function as the ones considered in the infimum in (55). Then

$$v(t,x) \leq \int_0^t L\big(\dot{\overline{w}}(s)\big)ds + g(y) = tH^*\left(\frac{x-y}{t}\right) + g(y).$$

Since this is true for any fixed $y \in \mathbb{R}^n$, taking the infimum over $y \in \mathbb{R}^n$, we can conclude $v(t,x) \leq u(t,x)$.

We remain to prove that $u(t,x) \leq v(t,x)$.
Let now considering w absolutely continuous and such that $w(t) = x$, then since L is convex we can apply Jensen inequality (for a proof see e.g. the book of Evans, Appendix B , Theorem 2) and we get

$$H^*\left(\frac{1}{t}\int_0^t \dot{w}(s)ds\right) \leq \frac{1}{t}\int_0^t H^*\big(\dot{w}(s)\big)ds = \frac{1}{t}\int_0^t L\big(\dot{w}(s)\big)ds.$$

Since $w(0) = y$ and $w(t) = x$, the left-hand side gives exactly

$$H^* \left(\frac{x-y}{t} \right),$$

therefore it is immediate to conclude

$$u(t,x) \le g(y) + tH^* \left(\frac{x-y}{t} \right) \le \int_0^t L\big(\dot{w}(s)\big) ds + g(y).$$

Since the previous inequality is true for any w absolutely continuous with $w(t) = x$ and $w(0) = y$, taking the infimum among all such $w(\cdot)$,
we can conclude $u(t,x) \le v(t,x)$. □

Remark 4.2. Under (46), $u(0,x) = g(x)$, for any $x \in \mathbb{R}^n$ and for any $t > 0$.

Lemma 4.1. Under the assumptions (46) and assuming that

$$g : \mathbb{R}^n \to \mathbb{R} \quad \text{is Lipschitz continuous and bounded,} \tag{57}$$

then, for any $t > 0$ and $x \in \mathbb{R}^n$,

$$u(t,x) = \min_{y \in \mathbb{R}^n} \left[g(y) + tH^* \left(\frac{x-y}{t} \right) \right]. \tag{58}$$

We leave the proof as an exercise.

Remark 4.3. Assuming $H^*(0) = 0$, then $u(t,x) \le g(x)$. This property is very useful to estimate the radius where the minimum above is attained.
If $H^*(0) \neq 0$, we get the more general bound: $u(t,x) \le g(x) + tH^*(0)$.

Note that, since $f(x,a) = -a$ and $l(x,a) = L(a) = H^*(a)$ are continuous and do not depend on x, they both satisfy in a trivial way the assumptions required in the previous section. Then we can apply Theorems 3.1 and 3.2 and get that the Hopf-Lax formula (58) is the unique viscosity solution of (45), but the assumption of compactness for the set A. Lipschitz functions have bounded gradient at almost every points. This is not enough but we assume that such a bound is at any point so that, by Remark 4.1, we can apply the above Theorems and conclude the following result.

Theorem 4.2. Under assumptions (46) and (57), the Hopf-Lax formula (58) is the unique viscosity solution of the Cauchy problem (45).

You can find a rigourous proof of the above theorem in the book of Evans, Section 10.2 Theorem 1 and Section 10.3, Theorem 3.
The proofs are very similar to the ones that we have showed in the previous section.

The key point is to use the following *functional identity*:

$$u(t,x) = \min_{y\in\mathbb{R}^n} \left[u(s,y) + (t-s)H^*\left(\frac{x-y}{t-s}\right)\right], \tag{59}$$

for any $x \in \mathbb{R}^n$ and $0 \le s < t$.

For a proof, see Lemma 1, Section 3.3 in the book of Evans.

Remark 4.4. Assuming $H^*(0) = 0$, then by (59), it easily follows that the Hopf-Lax function is non increasing in $t > 0$.

Lemma 4.2. Under the assumptions (46) and $H^*(0) = 0$, if $g : \mathbb{R}^n \to \mathbb{R}$ continuous and bounded, then the Hopf-Lax formula (58) is locally Lipschitz continuous in $\mathbb{R}^n \times [0, +\infty)$.

Proof. Note that H^* convex then H^* locally Lipschitz continuous. We are going to use this property in order to prove the same property for the Hopf-Lax function w.r.t. x.
Let us fix $t \in [0,T]$ and $x, y \in K$ with K compact set and let $\overline{y} \in \mathbb{R}^n$ be a point where the infimum for $u(t,y)$ is attained. Then

$$\begin{aligned} u(t,x) - u(t,y) &\le g(\overline{y}) + tH^*\left(\frac{x-\overline{y}}{t}\right) - g(\overline{y}) - tH^*\left(\frac{y-\overline{y}}{t}\right) \\ &= t\left[H^*\left(\frac{x-\overline{y}}{t}\right) - H^*\left(\frac{y-\overline{y}}{t}\right)\right] \end{aligned}$$

where we know that $\overline{y} \in \overline{B_{R(t)}}(y)$.
Note that $\overline{y}$ depends on t, too. Then if we show that $\forall\, t \in [0,T]$ there exists a compact $K = K(T) =\subset \mathbb{R}^n$ such that $x, y, \overline{y} \in K$, using that H^* is locally Lipschitz, we can conclude

$$u(t,x) - u(t,y) \le L_K|x - \overline{y} - y + \overline{y}| = L_K|x-y|,$$

where $L_K = \mathrm{Lip}(H^*; K)$ is the Lipschitz constant of H^* in the compact set K. Remember that the constant L_K depends on K but also on $T > 0$.

Thus, in order to get the Lipschitz continuity in space, we remain just to prove that $x, y, \overline{y} \in K$ for some compact set K which depends just on T.
Let us introduce the function

$$G(r) := \inf_{p\,:\,|p|=r} H^*(p) = \min_{p\,:\,|p|=r} H^*(p).$$

There exists a point $p_r \in \mathbb{R}^n$ such that $|p_r| = r$ such that $H^*(p_r) = G(r)$.
Note that

$$\frac{G(r)}{r} = \frac{H^*(p_r)}{|p_r|} \to +\infty, \quad \text{as } r \to +\infty.$$

Therefore (by the definition of limit), for any $L > 0$ there exists $\widetilde{r} = \widetilde{r}(L) > 0$ such that $\frac{G(r)}{r} > L$, for all $r > \widetilde{r}$.
Now let us fix $R > 1$ such that $\frac{R}{T} > \widetilde{r}$. Then for $|x - y| > R$, we get $\frac{|x-y|}{t} > \frac{R}{T} > \widetilde{r}$, which implies

$$tH^*\left(\frac{x-y}{t}\right) = \frac{H^*\left(\frac{x-y}{t}\right)}{\frac{|x-y|}{t}}\,|x-y| \geq \frac{G\left(\frac{|x-y|}{t}\right)}{\frac{|x-y|}{t}}\,|x-y| \geq L\,R > L, \tag{60}$$

since we have chosen $R > 1$.
Now let us consider $L = \overline{L} := 2\,\|g\|_\infty$ and $\overline{R} := \max\{1; \widetilde{r}(\overline{L})\}$.
Using (60) with $L = \overline{L}$ and $R = \overline{R}$, we get that

$$u(t,x) \geq g(x) + 2\,\|g\|_\infty \geq -\,\|g\|_\infty + 2\,\|g\|_\infty = \|g\|_\infty\,.$$

Since we know that $u(t,x) \leq g(x)$ (by the assumption $H^*(0) = 0$), then $u(t,x) \leq \|g\|_\infty$, therefore the infimum in $u(t,y)$ has to be attained in the closed ball $\overline{B_{\overline{R}}(y)}$, for any $t \in [0,T]$, where $\overline{R}$ depends just on the datum $g(x)$ and on $T > 0$.

This shows the claim and therefore the Lipschitz continuity in space for the Hopf-Lax function.

To show the Lipschitz continuity in time, we use the Functional Identity (59).

First note that, by Remark 4.4, $u(t,x)$ is non-increasing in time.
Hence in order to get the local Lipschitz continuity in time, fixed $x \in K \subset \mathbb{R}^n$ compact and $t \in [0,T]$, it is sufficient to find a constant $C > 0$ (depending on T and K) such that $u(t,x) - u(s,x) \geq -C(t-s)$.

So let us fix $T \geq t > s \geq 0$ and let L_K be the local Lipschitz constant of the Hopf-Lax function $u(t,x)$ w.r.t. the space-variable on the compact set

K_T, where $K_T \subset \mathbb{R}^n$ is a compact set such that, for any $s \in [0,T]$ and for any $x \in K$, the minimum points $\overline{y}(s)$ realizing the infimum in the Functional Identity (59) belong to K_T.

$$u(t,x) = \min_{y\in\mathbb{R}^n}\left[(t-s)H^*\left(\frac{x-y}{t-s}\right) + u(s,y) - u(s,x)\right] + u(s,x)$$
$$\geq \min_{y\in K_T}\left[-L_K + (t-s)H^*\left(\frac{x-y}{t-s}\right)\right].$$

Setting $z = \frac{x-y}{t-s}$, we can write

$$u(t,x) \geq \min_{z\in\mathbb{R}^n}\left[-L_K(t-s)|z| + (t-s)H^*(z)\right] =$$
$$-(t-s)\max_{z\in\mathbb{R}^n}\big[-L_K|z| + H^*(z)\big] = -(t-s)\max_{w\in\overline{B_{L_K}(0)}}\max_{z\in\mathbb{R}^n}\big[w\cdot z - H^*(z)\big]$$
$$= -(t-s)\max_{w\in\overline{B_{L_K}(0)}}|H(w)| = -C(t-s),$$

where $C = \max_{w\in\overline{B_{L_K}(0)}}|H(w)| > 0$ which depends on H, K and T. Therefore we can conclude

$$u(t,x) - u(s,x) \geq -C(t-s),$$

which implies the local Lipschitz continuity in time, i.e.

$$|u(t,x) - u(s,x)| \leq C|t-s|,$$

□

Example 4.4. Let us consider the following Cauchy problem

$$\begin{cases} u_t + \dfrac{1}{\alpha}|Du|^\alpha = 0, & t>0,\ x\in\mathbb{R}^n, \\ u(0,x) = g(x), & x\in\mathbb{R}^n. \end{cases} \tag{61}$$

then, for any $\alpha > 1$ and $g : \mathbb{R}^n \to \mathbb{R}$ bounded and uniformly continuous, the unique viscosity solution is given by

$$u(t,x) = \inf_{y\in\mathbb{R}^n}\left[g(y) + \frac{|x-y|^\beta}{\beta\, t^{\beta-1}}\right]. \tag{62}$$

Definition 4.2. In the particular case $\alpha = \beta = 2$, formula (62) is known as *inf-convolution* of the function $g(x)$. We will study in more details this particular case, in the next sections.

To conclude the section, we would like to quote the following more general theorem, which we are going to use in the exercise-session.

Theorem 4.3 (Alvarez -Barron-Ishii, 1999)**.** *Let us consider the Cauchy problem* (45)*, assuming $p \mapsto H(p)$ continuous and convex and such that the coercivity property* (46) *holds. Let be $g : \mathbb{R}^n \to \mathbb{R}$ continuous and such that*

$$g(x) \geq -C\big(|x| + 1\big), \tag{63}$$

for some $C > 1$, then the (unique) viscosity solution of (45) *if given by the Hopf-Lax formula* (58)*.*

Lemma 4.3. Under (63), show that there exists $C' > 0$ such that the following holds for the Hopf-Lax formula (58):

$$u(t,x) \geq -C'\big(|x| + t + 1\big).$$

Proof. First note that, for any $R > 0$, you have:

$$\begin{aligned} tH^*\left(\frac{x-y}{t}\right) &= \sup_{z\in\mathbb{R}^n} [z \cdot (x-y) - tH(z)] \\ &\geq \max_{z\in\overline{B_R}(0)} [z \cdot (x-y) - tH(z)] \geq \max_{z\in\overline{B_R}(0)} \left[z \cdot (x-y) - t \max_{z\in\overline{B_R}(0)} H(z)\right] \end{aligned}$$

Let $C(R) := \max_{z\in\overline{B_R}(0)} H(z)$, then

$$\begin{aligned} u(t,x) \geq \inf_{y\in\mathbb{R}^n} &\left[-C - C\,|y| + \max_{z\in\overline{B_R}(0)} \big[z \cdot (x-y)\big] - tC(R)\right] \\ &= \inf_{y\in\mathbb{R}^n} [-C - C\,|y| + R\,|x-y| - tC(R)] \\ &\geq \inf_{y\in\mathbb{R}^n} [-C - C\,|x| - C\,|x-y| + R\,|x-y| - tC(R)]\,. \end{aligned}$$

Choosing $R = C$, one can conclude. □

In the Exercises we will show further properties of the Hopf-Lax function.

5 Convexity and semiconvexity

5.1 Viscosity characterization of convex functions.

Let us start recalling the definition of convexity in $\mathbb{R}^n$.

Definition 5.1 (Convexity)**.** We say that a function $u : \mathbb{R}^n \to \mathbb{R}$ is *convex*, if for any $x, y \in \mathbb{R}^n$ and for any $\lambda \in (0,1)$, then

$$u(\lambda x + (1-\lambda)y) \le \lambda u(x) + (1-\lambda)u(y). \tag{64}$$

A continuous function u is concave if $-u$ is convex.

We remind that a $n \times n$-matrix M is non-negative definite, whenever

$$a^T M a = \big\langle Ma, a \big\rangle \ge 0, \quad \forall\, a \in \mathbb{R}^n.$$

It is well-known that any matrix M can be written as $M = S + A$,
where S is symmetric while A is antisymmetric.
Note: $a^T M a = a^T S a$, for any $a \in \mathbb{R}^n$.
Moreover, given any $n \times n$-matrix M, this is non-negative definite if and only if the minimum eigenvalue of the symmetric part of M is non-negative.

Lemma 5.1. Let be $u \in C^2(\mathbb{R}^n)$, then u is convex in $\mathbb{R}^n$ if and only if $D^2u(x) \ge 0$, for any $x \in \mathbb{R}^n$.

Proof. Let us assume that $u \in C^2(\mathbb{R}^n)$ is convex; in particular

$$u\left(\frac{y_1 + y_2}{2}\right) \le \frac{u(y_1) + u(y_2)}{2}, \quad \forall\, y_1, y_2 \in \mathbb{R}^n.$$

Let us fix $x, a \in \mathbb{R}^n$ and $\varepsilon > 0$, we can write the previous inequality for $y_1 = x + \varepsilon a$ and $y_2 = x - \varepsilon a$, which means

$$\begin{aligned} u(x) = u\left(\frac{(x+\varepsilon a) + (x - \varepsilon a)}{2}\right) &\le \frac{u(x+\varepsilon a) + u(x-\varepsilon a)}{2} \\ \le \frac{u(x) + \varepsilon Du(x)\cdot a + \varepsilon^2 a^T D^2u(x)a + u(x) - \varepsilon Du(x)\cdot a + \varepsilon^2 a^T D^2 u(x) a + o(\varepsilon^2)}{2} \\ &= u(x) + \varepsilon^2 a^T D^2 u(x) a + o(\varepsilon^2), \end{aligned} \tag{65}$$

where we have used the Taylor's expansion of order 2 centered at x for both $u(y_1)$ and $u(y_1)$. Hence by (65), we can deduce

$$\varepsilon^2 a^T D^2 u(x) a + o(\varepsilon^2) \ge 0,$$

which implies $a^T D^2u(x)a \geq 0$, just dividing by ε^2 and passing to the limit as $\varepsilon \to 0^+$. This concludes one implication.

To show the reverse implication, we first assume that the implication is true in the 1-dimensional case. A way to prove this claim is to use the geometric properties for 1-dimensional convex functions w.r.t. the secant and the Mean Value Theorem.

Assuming the result for $n = 1$, it is easy to show the same for $n > 1$.
In fact, let us assume that $D^2u(x) \geq 0$: we want to show that u is convex, which is equivalent to prove that, for any $x, y \in \mathbb{R}^n$, the function

$$G(\lambda) := u(\lambda x + (1-\lambda)y) - \lambda u(x) - (1-\lambda)u(y)$$

is non-positive in $[0,1]$.
First note that $G(0) = G(1) = 0$. Moreover, set $z := \lambda x + (1-\lambda)y$, then

$$G''(\lambda) = (x-y)^T D^2u(z)(x-y) \geq 0, \quad \text{for any } \lambda \in (0,1).$$

Hence $G(\lambda)$ is convex in $(0,1)$, which implies

$$G(\lambda) = G(\lambda\, 1 + (1-\lambda)\, 0) \leq \lambda G(1) + (1-\lambda)G(0) = 0,$$

which concludes the proof. □

A similar characterization holds for functions which are just continuous.

Definition 5.2 (Convexity in the viscosity sense)**.** We say that a continuous function is *convex in the viscosity sense* if

$$-D^2u(x) \leq 0, \quad \text{in the viscosity sense,} \tag{66}$$

which means

$$D^2\varphi(x_0) \geq 0,$$

for any $\varphi \in C^2(\mathbb{R}^n)$ such that $u - \varphi$ has a local maximum at x_0.

A continuous function is said *concave in the viscosity sense* if

$$-D^2u(x) \geq 0, \quad \text{in the viscosity sense,} \tag{67}$$

which means

$$D^2\varphi(x_0) \leq 0,$$

for any $\varphi \in C^2(\mathbb{R}^n)$ such that $u - \varphi$ has a local minimum at x_0.

Remark 5.1. A continuous function u is convex in the viscosity sense if and only if $-u$ is concave in the viscosity sense.

Theorem 5.1 (Alvarez-Lasry-Lions, 1997)**.** Let be $u \in C(\mathbb{R}^n)$, then u is convex if and only if u is convex in the viscosity sense.
The corresponding result holds for concave functions.

Proof: "u convex $\Longrightarrow$ u convex in the viscosity sense".
The proof is exactly the same that we have seen for the smooth case but applied to the test-functions.
Then let be $x_0 \in \mathbb{R}^n$ and $\varphi \in C^2$ such that $u(x_0) = \varphi(x_0)$ and $u(x) \leq \varphi(x)$ for x near x_0 and let us look at $y_1 = x_0 + \varepsilon a$ and $y_2 = x_0 - \varepsilon a$, with $a \in \mathbb{R}^n$ arbitrary and $\varepsilon > 0$ sufficiently small; the convexity of u implies

$$u(x_0) = u\left(\frac{y_1 + y_1}{2}\right) \leq \frac{u(y_1) + u(y_2)}{2} \leq \frac{\varphi(y_1) + \varphi(y_2)}{2}.$$

Applying the Taylor's expansion of order 2 centered at x_0 to the C^2-test function φ, respectively at the points y_1 and y_2, we can conclude. $\square$

Proof: "u convex in the viscosity sense $\Longrightarrow$ u convex".
This implication is much more complicate. We show this just for $n = 1$.
In the 1-dimensional case, the proof is very similar to the one already showed to characterize monotone functions, in the viscosity sense.
We argue by contradiction: let us suppose that there exist $t_1 < t_3$ and $\lambda \in (0, 1)$ such that

$$u(t_2) > \lambda u(t_1) + (1 - \lambda)u(t_3),$$

where $t_2 = \lambda t_1 + (1 - \lambda)t_3$. Note that it is possible to find $\varphi \in C^2$ with $\varphi'' < 0$ (i.e. strictly concave) in (t_1, t_3) such that

$$\begin{aligned} \varphi(t_1) &= u(t_1), \\ \varphi(t_3) &= u(t_3), \\ \varphi(t_2) &< u(t_2). \end{aligned}$$

Therefore there exists a $t \in (t_1, t_3)$ which is (local) maximum point for $u - \varphi$. Since by hypotesis $-u'' \leq 0$ in the viscosity sense, then $-\varphi''(t) \leq 0$ but this contradicts the fact that φ is a strictly concave function in (t_1, t_3). $\square$

Example 5.1. Note that $u(x) = |x|$ is convex in the viscosity sense in the whole $\mathbb{R}^n$. In fact, there are not test-functions touching u from above at 0, so the subsolution-property is trivially satisfied at 0 (try for exercise with $n = 1$). Moreover outside the origin the function is smooth and linear and therefore $D^2u(x) = 0$, for any $x \neq 0$.

5.2 Semiconcavity and semiconvexity

Definition 5.3 (Semiconcave and semiconvex functions)**.** Let be $u : \mathbb{R}^n \to \mathbb{R}$ continuous, then u is semiconcave if and only if there exists $C \geq 0$ such that

$$u(x+h) + u(x-h) - 2u(x) \leq C|h|^2, \quad \forall\, x, h \in \mathbb{R}^n. \tag{68}$$

The constant C is called semiconcavity constant of u.
A continuous function u is semiconvex if $-u$ is semiconcave.

Proposition 5.1 (Characterization of semiconcave functions.)**.** Given a continuous function $u : \mathbb{R}^n \to \mathbb{R}$, then the following are equivalent:

(i) u is semiconcave with constant $C \geq 0$ (i.e. (68) holds);

(ii) For any $x, y \in \mathbb{R}^n$ and for any $\lambda \in (0,1)$, it holds:

$$\lambda u(x) + (1-\lambda)u(y) + u(\lambda x + (1-\lambda)y) \leq C\frac{\lambda(1-\lambda)}{2}|x-y|^2; \tag{69}$$

(iii) The function $v(x) := u(x) - \frac{C}{2}|x|^2$ is concave;

(iv) The following viscosity inequality holds:

$$-D^2u(x) + C\, I_n \geq 0, \quad \text{in the viscosity sense.} \tag{70}$$

where by I_n we indicate the $n \times n$-identity-matrix.

Proof. Let us first show that (i) is equivalent (iii). Let us assume (i), an easy calculation shows that:

$$\begin{aligned} v(x+h) + v(x-h) - 2v(x) &= u(x+h) + u(x-h) - 2u(x) + \\ &- \frac{C}{2}\left(|x+h|^2 + |x-h|^2 - 2|x|^2\right) \leq C|h|^2 - C|h|^2 = 0. \end{aligned} \tag{71}$$

The inequality (71) is equivalent to the concavity property (see Proposition A.1.2. in the book of Cannarsa and Sinestrari "Semiconcave Functions, Hamilton-Jacobi Equations, and Optimal Control").
The reverse implication is exactly the same, therefore (i) are (iii) equivalent.

The equivalence between (iii) and (iv) follows directly by Theorem 5.1.

We remain to prove the equivalence between (ii) and (iii), which come from the identity

$$\lambda|x|^2 + (1-\lambda)|y|^2 + |\lambda x + (1-\lambda)y|^2 = \lambda.$$

Therefore an easy calculation shows that (ii) and the concavity for $v(x) = u(x) - \frac{C}{2}|x|^2$ are equivalent. □

Example 5.2. Let be $u(x) = |x|$, then u is not semiconcave at the origin. In fact (70) does not hold for any $C > 0$ at $x = 0$. Let be $C > 0$ and $\varphi(x) = C|x|^2$. Note that $u - \varphi$ has a local minimum at 0; in fact $u(0) = \varphi(0)$ and, let us choose $|x| \leq \frac{1}{C}$, then $\varphi(x) = C|x|^2 \leq C\,\frac{1}{C}\,|x| = u(x)$.
To conclude is sufficient to remark that $D^2\varphi(x) = 2C\,I_n > C\,I_n$ and this implies that u is not semiconcave at 0.

Example 5.3. Let be $u(x) = |x|^2 \in C^2(\mathbb{R}^n)$ and $D^2u(x) = 2I_n$ at any point $x \in \mathbb{R}^n$. Therefore u is both semiconcave and semiconvex.

Remark 5.2. Any C^2-function is both semiconcave and semiconvex in any bounded domain in $\mathbb{R}^n$. Moreover if a function is both semiconcave and semiconvex, one can always assume that the semiconvexity constant and semiconcavity constant are the same.

The following result states, in some sense, a reverse implication for the above remark.

Theorem 5.2. *[Cannarsa-Sinestrari, "Semiconcave Functions, Hamilton-Jacobi Equations, and Optimal Control" Section 3, Corollary 3.3.8.]*
If $u : \mathbb{R}^n \to \mathbb{R}$ is both semiconvex and semiconcave with constant $C \geq 0$, then $u \in C^{1,1}(\mathbb{R}^n)$ and the Lipschitz constant of Du is equal to $C \geq 0$.

Remark 5.3. Theorem 5.2 implies in particular that any function, which is semiconcave and semiconvex, has continuous first-order derivatives but it is also twice-differentiable almost everywhere.

Proposition 5.2 (Semiconcavity property of the Hopf-Lax formula)**.**
Let assume that $g : \mathbb{R}^n \to \mathbb{R}$ is bounded continuous and semiconcave with constant $C \geq 0$. The the Hopf-Lax formula $u(t,x)$ given by (58) is semiconcave with the same constant $C \geq 0$.

Proof. The proof is immediate. Let us consider a point $\overline{y} \in \mathbb{R}^n$ such that $u(t,x) = g(\overline{y}) + tH^*\left(\frac{x-\overline{y}}{t}\right)$, then choosing $y = \overline{y} + h$ and $y = \overline{y} - h$ to estimate respectively $u(t, x+h)$ and $u(t, x-h)$, one can deduce:

$$
\begin{aligned}
& u(t,x+h) + u(t,x-h) - 2u(t,x) \\
\leq\ & g(\overline{y}+h) + tH^*\left(\frac{x-\overline{y}}{t}\right) + g(\overline{y}-h) + tH^*\left(\frac{x-\overline{y}}{t}\right) - 2g(\overline{y}) - 2tH^*\left(\frac{x-\overline{y}}{t}\right) \\
& = g(\overline{y}+h) + g(\overline{y}-h) + -2g(\overline{y}) \leq C|h|^2.
\end{aligned}
$$

□

We conclude this section observing that all the above results can be state also for functions defined in convex bounded domains of $\mathbb{R}^n$.

5.3 Application to inf-convolution and sup-convolution.

Here we apply the results showed in the previous section to the particular case of the inf-convolution and the sup-convolution.

Given a function $g : \mathbb{R}^n \to \mathbb{R}$ bounded and continuous, we remind that the inf-convolution is the particular case of the Hopf-Lax formula (62) with $\beta = 2$, where the time-variable is considered as a parameter.
Let us give the definitions more precisely.

Definition 5.4 (inf-convolution)**.** For any $\varepsilon > 0$, the *inf-convolution* of the function g is defined as:

$$g_\varepsilon(x) = \inf_{y\in\mathbb{R}^n} \left[g(y) + \frac{|x-y|^2}{2\varepsilon} \right]. \tag{72}$$

Analogously one can introduce the sup-convolution of the function g.

Definition 5.5 (sup-convolution)**.** For any $\delta > 0$, the *sup-convolution* of the function g is defined as:

$$g^\delta(x) = \sup_{y\in\mathbb{R}^n} \left[g(y) - \frac{|x-y|^2}{2\delta} \right]. \tag{73}$$

The next Lemma shows that the inf-convolution and sup-convolution are approximations of the function g and states some their basic properties.

Lemma 5.2. Given a continuous function $g : \mathbb{R}^n \to \mathbb{R}$,
the following properties are true:

(i) If $g(x)$ is bounded by a constant G, then $g_\varepsilon(x)$ and $g^\delta(x)$ are both bounded by the same constant G. Moreover they are both locally Lipschitz.

(ii) If $g(x)$ is bounded then the infimum in (72) and the supremum in (73) are both attained in some points y_ε and y^δ which, respectively, belong to the balls $B_{\sqrt{4\,\|g\|_\infty\,\varepsilon}}(x)$ and $B_{\sqrt{4\,\|g\|_\infty\,\delta}}(x)$.

(iii) If $g(x)$ is Lipschitz with constant L, then $g_\varepsilon(x)$ and $g^\delta(x)$ are both Lipschitz with the same constant.

(iv) $g_\varepsilon(x)$ and $g^\delta(x)$ are both approximations of g, i.e.

$$\lim_{\varepsilon\to 0^+} g_\varepsilon(x) = g(x), \quad \text{and} \quad \lim_{\delta\to 0^+} g^\delta(x) = g(x).$$

Proof. One can prove the above properties exactly as we have done for the corresponding properties for the Hopf-Lax formula. So we leave those by exercise. □

The main result we are going to prove for the inf-convolution and sup-convolution is that, applying twice these regularizations to any bounded and continuous function g, one can get a $C^{1,1}$-approximation of g.
The idea is to show that the following functions are both semiconcave and semiconvex, for a suitable choice of the parameters $\varepsilon > 0$ and $\delta > 0$.

$$(g^\delta(x))_\varepsilon = \inf_{y\in\mathbb{R}^n}\left[g^\delta(y) + \frac{|x-y|^2}{2\varepsilon}\right] = \inf_{y\in\mathbb{R}^n}\sup_{z\in\mathbb{R}^n}\left[g(z) - \frac{|y-z|^2}{2\delta} + \frac{|x-y|^2}{2\varepsilon}\right],$$
$$(g_\delta(x))^\varepsilon = \sup_{y\in\mathbb{R}^n}\left[g_\delta(y) - \frac{|x-y|^2}{2\varepsilon}\right] = \sup_{y\in\mathbb{R}^n}\inf_{z\in\mathbb{R}^n}\left[g(z) + \frac{|y-z|^2}{2\delta} - \frac{|x-y|^2}{2\varepsilon}\right].$$

The following properties hold also for the Hopf-Lax formula, under suitable assumptions on the Hamiltonian H and it follows immediately by the stability of viscosity supersolutions and viscosity subsolutions, respectively, w.r.t. the operation of infimum and supremum.

Lemma 5.3. Let be $\mathcal{F}$ a family of continuous functions $v : \mathbb{R}^n \to \mathbb{R}$.

(i) If for any $v \in \mathcal{F}$, $v(x)$ is semiconvex with semiconvexity constant $C \geq 0$, then $u(x) := \sup_{v\in\mathcal{F}} v(x)$ is semiconvex with the same semiconvexity constant.

(i) If for any $v \in \mathcal{F}$, $v(x)$ is semiconcave with semiconcavity constant $C \geq 0$, then $u(x) := \sup_{v\in\mathcal{F}} v(x)$ is semiconcave with the same semiconcavity constant.

Proof. Let us show (i). We use the characterization of semiconvex functions by property (iv) of Proposition 5.1 and the stability of subsolution with respect to the operation of supremum (Proposition 2.3) and the result follows immediately. The other one is similar. □

Corollary 5.1. Given any initial datum g, for any $\varepsilon > 0$ and $\delta > 0$, the inf-convolution $g_\varepsilon(x)$ is semiconcave with semiconcavity constant $1/\varepsilon$ while the sup-convolution $g^\delta(x)$ is semiconvex with semiconvexity constant $1/\delta$.

Proof. For any fixed $\varepsilon > 0, \delta > 0$ and $y \in \mathbb{R}^n$, setting

$$F_1^y(x) := g(y) + \frac{|x-y|^2}{2\varepsilon} \quad \text{and} \quad F_2^y(x) := g(y) - \frac{|x-y|^2}{2\delta},$$

note that
$$D^2F_1^y(x) = \frac{1}{\varepsilon}I_n \quad \text{and} \quad D^2F_2^y(x) = -\frac{1}{\delta}I_n,$$
where I_n is the identity-$n \times n$-matrix.
Therefore $F_1^y(x)$ and $F_2^y(x)$ are both semiconvex and semiconcave for any $y \in \mathbb{R}^n$. By applying Lemma 5.3 we can conclude the corollary. □

Now we need to prove the following theorem. For sake of simplicity, let us first define:
$$g_\Phi(x) := \inf_{y\in\mathbb{R}^n} \big[g(y) + \Phi(x-y)\big]. \tag{74}$$

Definition 5.6. A function $\Phi(x)$ is *uniformly convex* if there exists $C > 0$ such that $\Phi(x) - \frac{C}{2}|x|^2$ is convex.

Whenever $\Phi \in C^2(\mathbb{R}^n)$, to be uniformly continuous is equivalent to require that $D^2\Phi(x) \geq C > 0$ while if the function is just continuous to be uniformly convex is equivalent to $-D^2\Phi(x) \leq -C < 0$, in the viscosity sense.

Remark 5.4. Note that in the case of the inf-convolution, we have
$$\Phi(x) = \frac{|x|^2}{2\varepsilon}$$
therefore on this case Φ is uniformly convex with constant $C = \frac{1}{\varepsilon}$.

Theorem 5.3. Let $g : \mathbb{R}^n \to \mathbb{R}$ continuous and bounded. Let us assume that Φ is uniformly convex w.r.t. a constant $C > 0$ and that g is semiconvex with semiconvexity constant $B \leq 0$.
If $B < C$, then g_Φ is semiconvex with constant $\widetilde{C} := \frac{BC}{C-B}$.

Remark 5.5. An analogous result holds for $g^\Phi = \sup_{y\in\mathbb{R}^n}\big[g(y) + \Phi(x-y)\big]$, assuming Φ uniformly concave and g semiconcave.
This case can be applied to the sup-convolution where $\Phi(x) = -\frac{|x|^2}{2\delta}$ is a uniformly concave function with constant $C = \frac{1}{\delta}$.

Before proving the previous theorem, we are going to show how to apply this to get the $C^{1,1}$-regularity for the functions $(g^\delta(x))_\varepsilon$ and $(g_\delta(x))^\varepsilon$.

Corollary 5.2. Given $g : \mathbb{R}^n \to \mathbb{R}$ continuous and bounded,
whenever $\delta > \varepsilon > 0$, the functions $\big(g^\delta(x)\big)_\varepsilon$ and $\big(g_\delta(x)\big)^\varepsilon$ are $C^{1,1}(\mathbb{R}^n)$.

Proof. Note first that, by Corollary 5.1 $g^\delta(x)$ is semiconvex with semiconvexity constant $B = \frac{1}{\delta}$. Moreover, Remark 5.4 tells that $\Phi(x) = \frac{|x|^2}{2\varepsilon}$ is uniformly convex with constant $C = \frac{1}{\varepsilon}$. Therefore, whenever $\frac{1}{\delta} < \frac{1}{\varepsilon}$, i.e. $\delta > \varepsilon$, we can

apply Theorem 5.3 and we get that $\left(g^\delta(x)\right)_\varepsilon$ is semiconvex.
Using again Corollary 5.1, we know also that $\left(g^\delta(x)\right)_\varepsilon$ is semiconcave since it is an inf-convolution. Then Theorem 5.2 implies that $\left(g^\delta(x)\right)_\varepsilon \in C^{1,1}(\mathbb{R}^n)$.
Similarly one can proceed to show that $\left(g_\delta(x)\right)^\varepsilon \in C^{1,1}(\mathbb{R}^n)$. □

Example 5.4. In the Exercises 9.21 and 9.22, we give some explicit examples of the previous result.

In particular we show that, starting from the function $g(x) = |x| \notin C^1(\mathbb{R}^n)$ or from the $g(x) = -|x| \notin C^1(\mathbb{R}^n)$, if $\delta > \varepsilon > 0$, then the sup-inf-convolution $\left(g_\delta(x)\right)^\varepsilon$ is $C^{1,1}(\mathbb{R}^n)$ (but in general not $C^2(\mathbb{R}^n)$).

To conclude we remain just to prove Theorem 5.3.

Proof Theorem 5.3. Note that, by setting $x_1 = x + h$ and $x_2 = x - h$ in (68), to prove that $g_\Phi(x)$ is semiconvex is equivalent to prove that, for any $x_1, x_2 \in \mathbb{R}^n$,

$$g_\Phi(x_1) + g_\Phi(x_2) - 2g_\Phi\left(\frac{x_1 + x_2}{2}\right) \geq -\widetilde{C}\left|\frac{x_1 - x_2}{2}\right|^2 \tag{75}$$

for some $\widetilde{C} > 0$.

To estimate $g_\Phi(x_1) + g_\Phi(x_2) - 2g_\Phi\left(\frac{x_1+x_2}{2}\right)$ from below we choose the two points y_1 and y_2, realizing respectively the infimum in (74) with $x = x_1$ and $x = x_2$. Moreover note that

$$-2g_\Phi\left(\frac{x_1 + x_2}{2}\right) \geq g(y) + \Phi\left(\frac{x_1 + x_2}{2} - y\right), \quad \forall\, y \in \mathbb{R}^n.$$

So in particular we can choose $y = \frac{y_1+y_2}{2}$ and we get

$$g_\Phi(x_1)+g_\Phi(x_2)-2g_\Phi\left(\frac{x_1 + x_2}{2}\right) \geq \Phi(x_1-y_1)+\Phi(x_2-y_2)-2\Phi\left(\frac{x_1 + x_2}{2} - \frac{y_1 + y_2}{2}\right)$$
$$+\, g(y_1) + g(y_2) - 2g\left(\frac{y_1 + y_2}{2}\right) =: I_\Phi + I_g, \tag{76}$$

where I_Φ is the sum of the first three terms on the right-hand side while I_g is the sum of the first three ones.

Let us first estimate I_Φ.
We claim that, by the uniform convexity assumption on Φ, we have

$$I_\Phi \geq \frac{C}{2}\left(|x_1 - y_1|^2 + |x_2 - y_2|^2 - 2\left|\frac{x_1 - y_1}{2} - \frac{x_2 - y_2}{2}\right|^2\right). \tag{77}$$

To check the claim (77), note that

$$\frac{x_1 + x_2}{2} - \frac{y_1 + y_2}{2} = \frac{x_1 - y_1}{2} + \frac{x_2 - y_2}{2}$$

and then write the definition of convexity for the function $\Phi(x) - \frac{C}{2}|x|^2$ in such mid-point.

Then by using the following identity

$$\begin{aligned}
&\frac{C}{2}\left(|x_1 - y_1|^2 + |x_2 - y_2|^2 - 2\left|\frac{x_1 - y_1}{2} - \frac{x_2 - y_2}{2}\right|^2\right) \\
&\qquad = \frac{C}{4}\left(|x_1 - x_2|^2 + |y_1 - y_2|^2 - 2\langle x_2 - x_1, y_2 - y_1\rangle\right) \\
&\qquad \geq \frac{C}{4}\left(|x_1 - x_2|^2 + |y_1 - y_2|^2 - \frac{C}{C-B}|x_1 - x_2|^2 - \frac{C-B}{C}|y_1 - y_2|^2\right) \\
&\qquad = \frac{BC}{C-B}|x_1 - x_2|^2 + B|y_1 - y_2|^2, \qquad (78)
\end{aligned}$$

where we have used that $\langle x_2 - x_1, y_2 - y_1\rangle = \lambda|x_2 - x_1|^2 + \frac{1}{\lambda}|y_2 - y_1|^2$, for any $\lambda > 0$ and we have chosen $\lambda = \frac{C}{C-B}$ which is positive and well-defined since the assumption $C > B \geq 0$.

Using (78) in (77), we can deduce

$$I_\Phi \geq \frac{BC}{C-B}|x_1 - x_2|^2 + B|y_1 - y_2|^2.$$

We remain to estimate I_g using the semiconvexity assumption on g, which tells:

$$I_g \geq -B|y_1 - y_2|^2,$$

hence, by (76), we conclude

$$\begin{aligned}
&g_\Phi(x_1) + g_\Phi(x_2) - 2g_\Phi\left(\frac{x_1 + x_2}{2}\right) \geq I_\Phi + I_g \\
&\qquad \geq \frac{BC}{C-B}|x_1 - x_2|^2 + B|y_1 - y_2|^2 - B|y_1 - y_2|^2 = \frac{BC}{C-B}|x_1 - x_2|^2,
\end{aligned}$$

which gives exactly (75) with $\widetilde{C} = \frac{BC}{C-B} \geq 0$. □

6 Discontinuous viscosity solutions.

The theory of viscosity solutions can be applied to discontinuous functions, too. We already remarked that the necessary regularity to state the subsolution requirement is the upper semicontinuity since we have to work with maximum points. Analogously, the necessary regularity for the supersolution requirement is the lower semicontinuity since this condition is related to testing at minimum points. For the notion of lower and upper semicontinuous functions and some examples, see the Appendix.
The previous remark is the key-point in order to introduce a definition of viscosity solutions for discontinuous functions.

Let us first recall the definition of upper and lower semicontinuous envelopes of a function $u : \mathbb{R}^n \to \mathbb{R}$, which are, respectively, given by

$$\begin{aligned} u^*(x) :&= \inf\{v(x) \,|\, v \geq u \text{ and } v \text{ lower semicontinuous}\} \\ &= \limsup_{\varepsilon \to 0^+} \{u(y) \,|\, |x-y| \leq \varepsilon\}, \end{aligned}$$

and

$$\begin{aligned} u_*(x) :&= \sup\{v(x) \,|\, v \leq u \text{ and } v \text{ upper semicontinuous}\} \\ &= \liminf_{\varepsilon \to 0^+} \{u(y) \,|\, |x-y| \leq \varepsilon\}. \end{aligned}$$

Definition 6.1. Let $u : \mathbb{R}^n \to \mathbb{R}$ be an a priori discontinuous function, then u is a (discontinuous) viscosity solution of

$$F(x, u(x), Du(x), D^2u(x)) = 0, \tag{79}$$

if and only if,

(i) u^* is a viscosity subsolution for the equation (79),

(ii) u_* is a viscosity supersolution for the equation (79).

Remark 6.1. It is immediate to note that if u is continuous, then the previous definition is exactly the standard definition of viscosity solution, (in fact: $u = u^* = u_*$ whenever u is a continuous function).

In the discontinuous case, many of the results shown in the continuous case, still hold.

More information on discontinuous viscosity solutions can be found in the book of Bardi and Capuzzo Dolcetta and in the book of Barles "Solutions de Viscosité des Équations de Hamilton- Jacobi" and in the paper of Barles "Discontinuous viscosity solutions of first-order Hamilton-Jacobi equations: a guided visit" (1993).

7 An example of degenerate elliptic PDE.

7.1 Elliptic and degenerate elliptic second-order PDEs.

A general theory for the viscosity solutions for second-order PDEs, under the assumptions (9) and (10), can be found in the "User's guide to viscosity solutions of second order partial differential equations" by Crandall, Ishii and Lions. Moreover we refer to the book of Caffarelli and Cabrè "Fully Nonlinear Elliptic Equations", too, where one can find a complete regularity theory for second-order, fully nonlinear, uniformly elliptic, PDEs.

We would like to recall that a nonlinear PDE is called *fully nonlinear* if it is nonlinear in the highest order part; e.g. $\Delta_\infty u = 0$ is fully nonlinear while $\Delta u + |Du| = 0$ is nonlinear but it is not fully nonlinear.

Moreover, assuming for sake of simplicity $F(D^2u(x)) = f(x)$, a second-order PDE is *uniformly elliptic* if there exist two constants $0 < \lambda \leq \Lambda$ (which are called constants of ellipticity) such that

$$\lambda \|N\| \leq F(M) - F(M+N) \leq \Lambda \|N\|, \quad \forall N \geq 0. \tag{80}$$

Note in particular that $\lambda \|N\| \geq 0$, which implies $F(M) \geq F(M+N)$.
Hence a uniformly elliptic PDE in particular satisfies assumption (9), too.

Here we are interested in understanding what happens if the function $F(x, z, p, M)$ is not continuous w.r.t the variables p and/or M (which means that the equation is not continuous w.r.t. to the first-order and/or the second-order derivatives).
Next we give the definition of viscosity solutions in this degenerate case. For more information on a theory of viscosity solutions for a large class of degenerate elliptic PDEs, we refer to the book of Giga "Surface Evolution Equations".

Note that, for sake of simplicity, we omit to consider the case of equation depending on u, i.e. we assume $F(x, z, p, M) = F(x, p, M)$.

Definition 7.1. Let us look at the equation

$$F(x, Du(x), D^2u(x)) = 0, \quad x \in \Omega, \tag{81}$$

with $F(x, p, M)$ a priori discontinuous w.r.t. the variables p and M.
Given a continuous function $u : \Omega \to \mathbb{R}$, we say that u is a viscosity solution of (81) if and only if

(i) u is a viscosity subsolution of the equation

$$F_*(x, Du(x), D^2u(x)) = 0, \quad x \in \Omega,$$

where

$$F_*(x,p,M) = \liminf_{\varepsilon\to 0^+} \big\{F(q,N) \,|\, |p-q| \le \varepsilon,\ \|N-M\| \le \varepsilon\big\};$$

(ii) u is a viscosity supersolution of the equation

$$F^*(x, Du(x), D^2u(x)) = 0, \quad x \in \Omega,$$

where

$$F^*(x,p,M) = \limsup_{\varepsilon\to 0^+} \big\{F(q,N) \,|\, |p-q| \le \varepsilon,\ \|N-M\| \le \varepsilon\big\}.$$

We recall that, if $u : \Omega \to \mathbb{R}$ is discontinuous on Ω, we require the property (i) for the upper semicontinuous envelope u^* and the property (ii) for the lower semicontinuous envelope u_*.

In particular, existence results and comparison principles hold for the viscosity solutions of the previous equation, whenever $F(x,p,M) = F(p,M)$ satisfies the degenerate elliptic assumption (9) and the following *geometric condition*:

$$F\big(\mu\, p, \mu\, M + \alpha\, p \otimes p\big) = \mu\, F(p,M), \tag{82}$$

for any $\mu > 0$, $\alpha \in \mathbb{R}$, $p \in \mathbb{R}^n$ with $p \neq 0$ and M symmetric $n \times n$-matrix, and where $p \otimes p$ is the $n \times n$ matrix defined by $p \otimes p := p\,p^T$.

We want also to point out that an analogous definition and similar existence-results and uniqueness-results can be given in the case of an evolution equation:

$$u_t + F(x, Du, D^2u) = 0,$$

with $F(x,p,M)$ degenerate elliptic.

In the next sections we are going to study a particular degenerate elliptic PDE: the level-set equation for the evolution by mean curvature flow.

7.2 The geometric evolution by mean curvature flow.

Let $\Sigma \subset \mathbb{R}^n$ be an hypersurface, i.e.dim $\Sigma = n-1$ (for sake of simplicty, one may think of the case $n = 3$, then Σ is a surface).
From now on, we assume $n \geq 2$.

Let us indicate by $\boldsymbol{n}(x)$ the *external normal* at the point $x \in \Sigma$, we recall the definition of *mean curvature* for a hypersurface $\Sigma \subset \mathbb{R}^n$, which is the scalar defined, at any point $x \in \Sigma$, as

$$k(x) := \operatorname{div}(\boldsymbol{n}(x)).$$

Remark 7.1. In geometry the mean curvature is defined as the mean of the principal curvatures at any point $x \in \Sigma$, i.e.

$$\widetilde{k}(x) := \frac{k_1(x) + \cdots + k_n(x)}{n},$$

where $k_1(x), \ldots, k_n(x)$ are the principal curvatures at the point x.
Note that $k(x) = n\,\widetilde{k}(x)$, so they differ just by a constant.

The *curvature vector* is given by

$$\boldsymbol{k}(x) := k(x)\boldsymbol{n}(x) = \operatorname{div}(\boldsymbol{n}(x))\boldsymbol{n}(x).$$

Example 7.1 (The sphere)**.** Let us consider a sphere $S_R^{n-1}(p) := \partial B_R(p)$, which is a hypersurface in $\mathbb{R}^n$. Without loss of generality, we may assume $p = 0$. Then the external normal, at any point x, is given by $\boldsymbol{n}(x) = \frac{x}{|x|}$ (recall that, if $x \in S_R^{n-1}(0)$, $|x| = R$).
So an easy calculation shows that

$$k(x) = \operatorname{div}(\boldsymbol{n}(x)) = \operatorname{div}\left(\frac{x}{|x|}\right) = \frac{n-1}{|x|} = \frac{n-1}{R}.$$

Exercise 7.1. For $R > 0$, calculate the mean curvature of the cylinder
$C_R := \{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 = R^2\}$.

Definition 7.2 (Evolution by mean curvature flow)**.** Given a hypersurface $\Sigma_0 \subset \mathbb{R}^n$, we say that a family of hypersurfaces $\{\Sigma(t)\}_{t \geq 0}$ is an *evolution by mean curvature flow* of the initial hypersurface Σ_0, if and only if

(i) $\Sigma(0) = \Sigma_0$;

(ii) Recalling that the normal velocity at a point $x(t)$ is defined as $\boldsymbol{v}^n(x(t)) = \big(\dot{x}(t) \cdot \boldsymbol{n}(x(t))\big)\boldsymbol{n}(x(t))$ then, for any $x(t) \in \Sigma(t)$,

$$\boldsymbol{v}^n(x(t)) = -\boldsymbol{k}(x(t)),$$

which means $\dot{x}(t) \cdot \boldsymbol{n}(x(t)) = -k(x(t))$.

Remark 7.2. The evolution by mean curvature flow is related to hypersurfaces which evolve trying to minimize the area. In fact the mean curvature flow is the gradient flow of the area-functional.

Let us now consider a particular family of evolving hypersurfaces: the hypersurfaces evolving without changing their shapes.

Definition 7.3 (Homotetic solutions)**.** A family of hypersurfaces $\{\Sigma(t)\}_{t\geq 0}$ is a *homotetic solution* (or also *self-similar solution*) of the evolution by mean curvature flow from an initial hypersurface Σ_0, if there exists a family of numbers $\lambda(t) > 0$ such that $\lambda(0) = 1$ and

$$\Sigma(t) := \lambda(t)\Sigma_0 = \{(x_1(t), \dots, x_n(t)) = (\lambda(t)\, x_1, \dots, \lambda(t)\, x_n) \,|\, (x_1, \dots, x_n) \in \Sigma_0\}$$

is an evolution by mean curvature flow of Σ_0.

Example 7.2 (The sphere)**.** The sphere is a homotetic solution (Fig. 1), therefore $\Sigma(t) = \partial B_{R(t)}(p)$. We assume that the initial hypersurface is $\Sigma_0 = \partial B_{R_0}(p)$, i.e. $R(0) = R_0$. The normal velocity is $\dot{R}(t)$, so if $\Sigma(t) \subset \mathbb{R}^n$ is an evolution by mean curvature flow of Σ_0, we can deduce the following differential equation for the radius $R(t)$:

$$\dot{R}(t) = -k(x(t)) = -\frac{n-1}{R(t)}, \quad \text{with } R(0) = R_0.$$

Solving the previous equation, we find

$$R(t) = \sqrt{R_0^2 - 2(n-1)t}\,.$$

Note that

$$R(\,\widehat{t}\,) = 0 \quad \Longleftrightarrow \quad \widehat{t} = \frac{R_0^2}{2(n-1)}.$$

When the radius $R(t)$ degenerates to 0, the hypersurfaces shrink in the center. There the evolution expires since a point is not anymore a hypersurface. The time $\widehat{t} > 0$ is called *expiration time*.

More in general we may introduce the following notion.

Definition 7.4. The *expiration time* is the time $\widehat{t} > 0$ satisfying

$$\dim\big(\Sigma(\,\widehat{t}\,)\big) \leq n-2 \quad \text{and} \quad \dim\big(\Sigma(t)\big) = n-1, \text{ for any } 0 \leq t < \widehat{t}.$$

At the expiration time the evolution expires since $\Sigma(\,\widehat{t}\,)$ is not anymore a hypersurface in $\mathbb{R}^n$.

Exercise 7.2. Assuming that the cylinder $C_R := \{(x,y,z) \in \mathbb{R}^3 \,|\, x^2 + y^2 = R^2\}$ is a homotetic solution of the evolution by mean curvature flow, find $\lambda(t)$ and the expiration time (Fig. 2).
Note that in this case the surface shrinks in a line and not in a point.

In the book of Ecker "Regularity Theory for Mean Curvature Flow", it is proved that, if we assume that $k(x(t)) > 0$ for any $x(t) \in \Sigma(t)$ and that $\Sigma(t)$ is a smooth hypersurface for any $t > 0$, then the unique homotetic solutions of the evolution by mean curvature flow are the *spherical cylinders*, which are hypersurfaces in $\mathbb{R}^n$ defined, for any $0 \leq k \leq n$, as

$$C_{n,k}(t) := \partial B^{n+1-k}_{R(t)} \times \mathbb{R}^k.$$

Note that if $k = 0$ then $C_{n,0}(t)$ is a sphere in $\mathbb{R}^n$ while, whenever $n = 3$ and $k = 1$, we get a classical cylinder in $\mathbb{R}^3$.

We conclude giving briefly some examples of surfaces evolving by mean curvature flow. More details (and nice pictures, too) can be found in the book of Ecker.

Example 7.3 (Tori)**.** A torus evolving by mean curvature flow evolves in tori (but not in self-similar tori) until it shrinks in a S^1 or in a point, depending on the relationship between the two radius defining the initial torus (i.e. depending "how much fat the initial torus is"), see Figures 3 and 4.

Example 7.4 (Convex sets)**.** A convex set in general evolves trying to get as much similar as possible to a sphere and then it shrinks, in finite times, in some internal point.

Problems can arise when the starting evolving hypersurface is not convex.

Example 7.5 (The dumbbell)**.** A dumbbell can build as two spheres (with equal radius) connected by a thin cylinder. Assuming that the two spheres are very big, while the cylinder is long enough and thin enough, it is easy to understand the corresponding evolution by mean curvature flow.
In fact, the cylinder shrinks very fast in a line while the two balls evolve very slowly in almost self-similar balls. Therefore after a while the dumbbell

shrinks in two disconnected balls. In that moment an meaningful topological change occurs and it is not possible to speak anymore of normal vector or curvature vector. Then the two spheres go on evolving separately until they shrink in two different internal points (see Fig. 5).

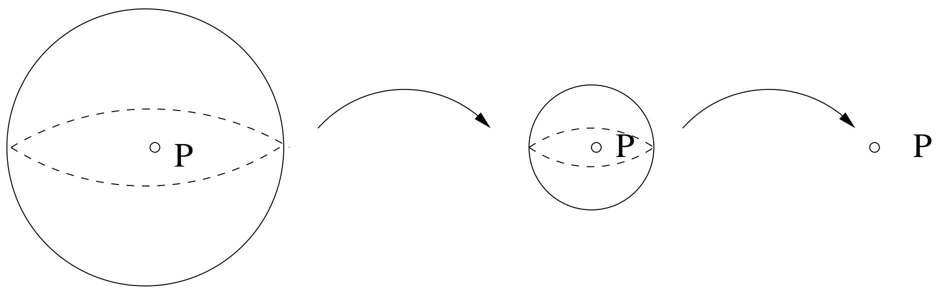


Figure 1: `The sphere evolving by mean curvature flow.`

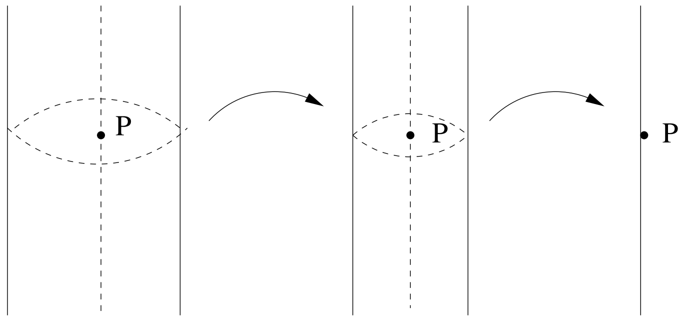


Figure 2: `The cylinder evolving by mean curvature flow.`

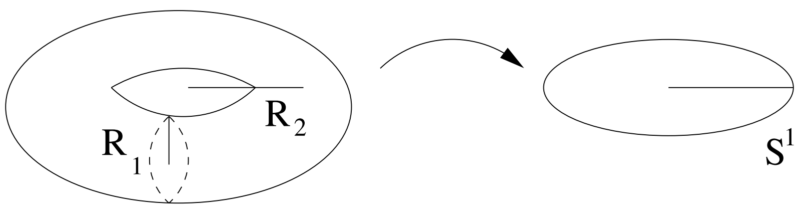


Figure 3:`The torus evolving by mean curvature flow with`$R_1 << R_2$.

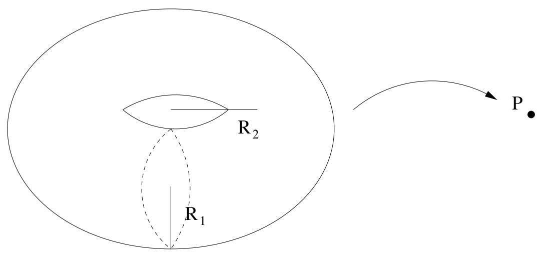


Figure 4: The torus evolving by mean curvature flow with $R_1 \approx R_2$.

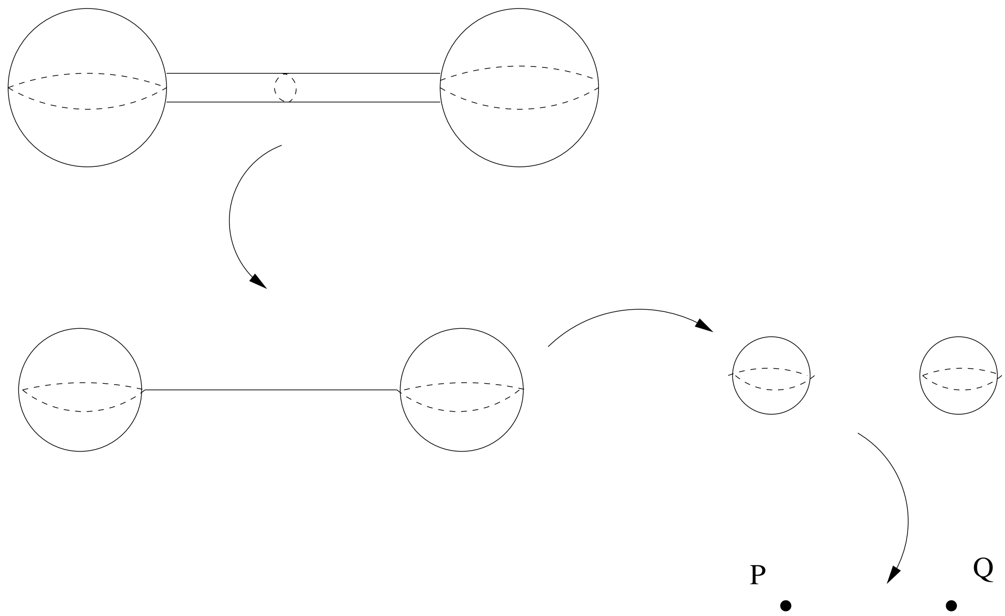


Figure 5: The dumbbell evolving by mean curvature flow.

The example of the dumbbell can be made easily in a smooth hypersurface and the corresponding evolution does not change significantly.
Therefore, even initial smooth hypersurfaces, evolving by mean curvature flow, can generate singularities in finite times.

To try to overcome this problem, several weak notions of evolution by mean curvature flow were introduced in the years. We are going to study the, so called, evolution by level-sets.

7.3 The level-set equation.

The level-set approach was introduced in 1991, independently, by Evans and Spruck and by Chen, Giga and Goto.

The idea is to write the initial hypersurface Σ_0 and the evolving family of hypersurfaces $\{\Sigma(t)\}_{t\geq 0}$ as level-sets of suitable functions, which means

$$\Sigma_0 = \{x \in \mathbb{R}^n \,|\, u_0(x) = 0\} \quad \text{and} \quad \Sigma(t) = \{x \in \mathbb{R}^n \,|\, u(t,x) = 0\},$$

for some $u_0 : \mathbb{R}^n \to \mathbb{R}$ and $u : [0, \widehat{t}\,] \to \mathbb{R}^n$ where $\widehat{t}$ is the expiration time of the evolution.

Assuming that $\Sigma(t)$ is an evolution by mean curvature flow of Σ_0, we derive the PDE solved by u.

Let us assume that all the hypersurfaces involved are smooth, we observe that $x(t) \in \Sigma(t)$ if and only if $u(t, x(t)) = 0$, for any $t \in [0, \widehat{t}\,]$, which implies

$$\frac{d}{dt} u(t, x(t)) = 0. \tag{83}$$

Note also that equation (83) is indeed equivalent to the fact: $x(t) \in \Sigma(t)$, if we consider every constant-level sets instead of just the zero-level sets.

Equation (83) is equivalent to

$$u_t + Du \cdot \dot{x}(t) = 0. \tag{84}$$

Since $\Sigma(t)$ are evolving by mean curvature flow, by Definition 7.2 we know that the normal velocity at any point is equal to minus the mean curvature at that point.

We recall that, whenever a hypersurface is defined as the zero-level set (or more in general the constant-level set) of some function, then the external normal direction is given by the renormalized gradient of the function at such a point, i.e.

$$\boldsymbol{n}(x) = \frac{Du(x)}{|Du(x)|}.$$

This gives the following computations:

$$\begin{aligned}
Du \cdot \dot{x}(t) &= -|Du|\,|\boldsymbol{v}^n(x(t))| = -|Du| k(x(t)) = -|Du| \mathrm{div}\left(\frac{Du}{|Du|}\right) \\
&= -|Du|\left(\frac{\mathrm{div}(Du)}{|Du|} - (Du)^T D^2 u \,\frac{Du}{|Du|^3}\right) = -\Delta u + \left\langle D^2 u \frac{Du}{|Du|}, \frac{Du}{|Du|}\right\rangle \\
&= -\Delta u + \Delta_\infty u,
\end{aligned} \tag{85}$$

where Δu is the standard Laplacian of u while $\Delta_\infty u$ is the, so called, infinite-Laplacian of u.

Using (85) in (84), we find that $u(t,x)$ solves the following PDE:

$$u_t - \Delta u + \Delta_\infty u = 0. \tag{86}$$

The previous nonlinear second-order PDE is called *level-set equation* for the evolution by mean curvature flow.

Next we introduce the following weak notion for the evolution by mean curvature flow, using the viscosity solutions of the level-set equation (86).

Definition 7.5. Given an initial continuous function $u_0 : \mathbb{R}^n \to \mathbb{R}$, we say that $\{\Sigma(t)\}_{t\geq 0}$ is a *generalized evolution by mean curvature flow* of the initial hypersurface $\Sigma_0 = \{x \in \mathbb{R}^n \,|\, u_0(x) = 0\}$ if and only if $\Sigma(t) = \{x \in \mathbb{R}^n \,|\, u(t,x) = 0\}$ where $u(t,x)$ is the (unique) viscosity solution of

$$\begin{cases} u_t - \Delta u + \Delta_\infty u = 0, & x \in \mathbb{R}^n, t > 0, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n. \end{cases} \tag{87}$$

Note that the PDE in (87) is degenerate whenever $|Du| = 0$ since the infinite-Laplacian operator is so. Thus, we understand the viscosity solutions of (87) by applying Definition 7.1.

Therefore we have to calculate the upper and lower envelopes $F^*(p,M)$ and $F_*(p,M)$ with

$$F(p,M) := -\mathrm{Tr}(M) + \left\langle M\frac{p}{|p|}, \frac{p}{|p|}\right\rangle = -\mathrm{Tr}\left(\left(Id - \frac{p\otimes p}{|p|^2}\right)M\right),$$

where by Id we mean the $n\times n$-identity matrix.

First we recall some useful properties for the trace of a matrix, which is defined as the sum of the elements on the diagonal:

(i) The trace of a scalar is the scalar itself, i.e.

$$\mathrm{Tr}(\lambda) = \lambda, \quad \forall\, \lambda \in \mathbb{R}.$$

(ii) The trace is a linear operator, i.e.

$$\mathrm{Tr}(\lambda\, A + \mu\, B) = \lambda\, \mathrm{Tr}(A) + \mu\, \mathrm{Tr}(B),$$

for all $\lambda, \mu \in \mathbb{R}$ and A and B $n \times n$-matrices.

(iii) The trace is invariant under permutations, i.e.

$$\mathrm{Tr}(ABC) = \mathrm{Tr}(BCA) = \mathrm{Tr}(CAB),$$

for all A, B and C $n \times n$-matrices.

(iv) The trace is invariant under orthonormal reparametrizations, i.e.

$$\mathrm{Tr}(O^T AO) = \mathrm{Tr}(A),$$

for any A $n \times n$-matrix and any O orthonormal $n \times n$-matrix (i.e. $O^T\, O = Id$). Note that this property is a trivial consequence of property (iii). Moreover, this implies that whenever the matrix M can be written in a diagonal form (e.g. whenever M is a real symmetric $n \times n$-matrix, as in our case), then the trace can be defined as the sum of the eigenvalues.

(v) For any non-negative definite matrices $M, N \geq 0$,

$$\mathrm{Tr}(M\, N) \geq 0.$$

Note that the trace of a non-negative definite matrix is always non-negative. Nevertheless, the fact that M and N are both non-negative does not in general imply that the product matrix $M\, N$ is non-negative, too. In general $M\, N \geq 0$ just in the case when M and N commute.

We remain to calculate F^* and F_*.

Note that, since only the infinite-Laplace operator is degenerate, it is enough to calculate the upper and lower regularizations of

$$G(p, M) = \mathrm{Tr}\left(\frac{p \otimes p}{|p|^2}\, M\right),$$

at the point $p = 0$. Using the above properties of the trace, it is not difficult to show that

$$G^*(0, M) = \lambda_{\max}(M),$$

while

$$G_*(0, M) = \lambda_{\min}(M),$$

where by $\lambda_{\max}(M)$ and $\lambda_{\min}(M)$ we mean the maximum and the minimum eigenvalues of the matrix M, respectively.
Hence we can give the following definition.

Definition 7.6. Let $\Sigma_0 = \{x \in \mathbb{R}^n | u_0(x) = 0\}$ be a hypersurface in $\mathbb{R}^n$.
We say that $\Sigma(t) = \{x \in \mathbb{R}^n | u(t, x) = 0\}$ is a *generalized evolution by horizontal mean curvature flow* if u is a continuous function satisfying $u(0, x) = u_0(x)$ and such that

1. for any $\varphi \in C^2(\mathbb{R}^n \times (0, +\infty))$ such that $u - \varphi$ has a local minimum at (t_0, x_0), then

$$\begin{cases} \varphi_t - \Delta\varphi + \Delta_\infty\varphi \geq 0, & \text{at } (t_0, x_0), \text{ if } D\varphi(t_0, x_0) \neq 0, \\ \varphi_t - \Delta\varphi + \lambda_{\max}(D^2\varphi) \geq 0, & \text{at } (t_0, x_0), \text{ if } D\varphi(t_0, x_0) = 0. \end{cases} \tag{88}$$

2. for any $\varphi \in C^2(\mathbb{R}^n \times (0, +\infty))$ such that $u - \varphi$ has a local maximum at (t_0, x_0), then

$$\begin{cases} \varphi_t - \Delta\varphi + \Delta_\infty\varphi \leq 0, & \text{at } (t_0, x_0), \text{ if } D\varphi(t_0, x_0) \neq 0, \\ \varphi_t - \Delta\varphi + \lambda_{\min}(D^2\varphi) \leq 0, & \text{at } (t_0, x_0), \text{ if } D\varphi(t_0, x_0) = 0. \end{cases} \tag{89}$$

Remark 7.3. Evans and Spruck use a slightly different definition of generalized evolution by mean curvature flow, but it is possible to show that the two definitions are indeed equivalent.

Remark 7.4. It is possible to show that the definition of generalized evolution by mean curvature flow does not depend on the chosen parametrization u_0 but just on the zero-level set: $\{u_0 = 0\}$. This means that, whenever $\{u_0 = 0\} = \{v_0 = 0\}$ then $\{u(t, x) = 0\} = \{v(t, x) = 0\}$, where u and v are the viscosity solutions of (87), with initial condition u_0 and v_0 respectively.
A proof of this result can be found in the book of Giga.
We just remark that the key point is a technical reparametrization of the function v_0 in order to be able to apply the comparison principles for the viscosity solutions of the level-set equation.

Now let us recall that the level-set equation for the evolution by mean curvature flow can be written as

$$u_t + F(Du, D^2u) = 0,$$

where

$$F(p, M) = -\mathrm{Tr}\left(\left(Id - \frac{p \otimes p}{|p|^2} \right) M \right). \tag{90}$$

We want to conclude proving that the level-set equation is a degenerate elliptic, geometric PDE.

Lemma 7.1. The level-set equation for the evolution by mean curvature flow is degenerate elliptic, i.e. $F(p, M)$ given in (90) satisfies property (9).

Proof. Let us first remark that, since the trace is linear, then property (9) is equivalent to

$$-\mathrm{Tr}\left(\left(Id - \frac{p \otimes p}{|p|^2} \right) \left(M - N \right) \right) \leq 0, \quad \forall\, M \geq N. \tag{91}$$

Note that $M - N \geq 0$, thus by the property (v) for the trace, in order to get (91), it is sufficient to prove that the matrix

$$B(p) := \left(Id - \frac{p \otimes p}{|p|^2} \right)$$

is non-negative definite, for any $p \neq 0$.
This means to show that, for any $\eta \in \mathbb{R}^n$, we have $\eta^T B(p)\, \eta \geq 0$.
Let us first estimate

$$\eta^T \frac{p \otimes p}{|p|^2} \eta = \frac{\eta^T p\, p^T \eta}{|p|^2} = \frac{(p^T \eta)^T\, p^T \eta}{|p|^2} = \frac{|p^T \eta|^2}{|p|^2} \leq \frac{\left|p^T\right|^2 |\eta|^2}{|p|^2} = \frac{|p|^2 |\eta|^2}{|p|^2} = |\eta|^2,$$

by Cauchy-Schwartz inequality.

Therefore, we can conclude

$$\eta^T B(p)\, \eta = \eta^T \eta - \eta^T \frac{p \otimes p}{|p|^2} \eta \geq |\eta|^2 - |\eta|^2 = 0,$$

which tells that $B(p) \geq 0$, for any $p \neq 0$, that implies (91) and so (9). □

Lemma 7.2. The level-set equation for the evolution by mean curvature flow is geometric, i.e. $F(p, M)$ given in (90) satisfies property (82).

Proof. Let us set $Y := \mu\, M + \alpha\, p \otimes p$ and $\widetilde{p} = \mu\, p$ where $p \neq 0$. We have to prove that

$$F(Y, \widetilde{p}) = \mu\, F(p, M). \tag{92}$$

We first calculate

$$\frac{p \otimes p}{|p|^2}\, p \otimes p = \frac{(p\, p^T)(p\, p^T)}{|p|^2} = \frac{p\,(p^T p)\, p^T}{|p|^2} = \frac{p\, |p|^2\, p^T}{|p|^2} = p \otimes p.$$

Hence, identity (92) is given by the following easy calculation:

$$\begin{aligned}
&F(Y, \widetilde{p}) \\
&= -\mathrm{Tr}\left(\left(Id - \frac{\widetilde{p} \otimes \widetilde{p}}{|\widetilde{p}|^2} \right) Y \right) = -\mathrm{Tr}\left(\left(Id - \frac{\mu\, p \otimes \mu\, p}{\mu^2\, |p|^2} \right) \big(\mu\, M + \alpha\, p \otimes p \big) \right) \\
&= -\mathrm{Tr}\left(\mu\, M + \alpha\, p \otimes p - \mu \frac{p \otimes p}{|p|^2} M - \alpha \frac{p \otimes p}{|p|^2}\, p \otimes p \right) \\
&= -\mathrm{Tr}\left(\mu\, M + \alpha\, p \otimes p - \mu \frac{p \otimes p}{|p|^2} M - \alpha\, p \otimes p \right) \\
&= -\mathrm{Tr}\left(\mu\, M - \mu \frac{p \otimes p}{|p|^2} M \right) \\
&= -\mu\ \mathrm{Tr}\left(\left(Id - \frac{p \otimes p}{|p|^2} \right) M \right) \\
&= \mu\, F(p, M).
\end{aligned}$$

□

One can prove the existence by Perron's method and comparison principles for the generalized evolution by mean curvature flow, as a particular case of (evolution) degenerate elliptic PDEs under the geometric assumption (82) (e.g. see the book of Giga for details on these results).

In the particular case of the level-set equation, direct proofs for existence and uniqueness of viscosity solutions can be found also in the already quoted papers by Evans and Spruck and by Chen,Giga and Goto, where these ideas where first introduced.

7.4 The Kohn-Serfaty game.

In this section we present an approximation-method for the viscosity solutions of the level-set equation of the evolution by mean curvature flow, by using a suitable deterministic two-person game. This game and the related results have been proved by Kohn and Serfaty in "A deterministic-control-based approach to motion by mean curvature", published in Communication in Pure and Applied Mathematics (2005).

Let us consider $\Omega \subset \mathbb{R}^2$ open, bounded and convex and let us fix a constant $\varepsilon > 0$ which is called *step of the game*.

The game consists in two players, which we will call Player I and Player II, playing against each others.
The rules of the game are pretty easy:

1. Player I chooses a direction, which means that he chooses a vector $\boldsymbol{v} \in \mathbb{R}^2$ such that $\|\boldsymbol{v}\| = 1$ (to make the mathematical techniques working easier, one may assume $\|\boldsymbol{v}\| \leq 1$).
2. Player II chooses a scalar $b = \pm 1$.

The game evolves from a point $x \in \Omega$ to a point $y := x + \sqrt{2}\,\varepsilon\,\boldsymbol{v}$.
Note that y does not necessary belong to Ω.

The goal of Player I is to go out from the set, i.e. equivalently to reach $\partial\Omega$, while the goal of Player II is to keep the game inside the open domain Ω. When the game reaches (or crosses) a point of $\partial\Omega$, the game ends and Player I wins. Of course Player I wants to win as soon as possible, so he wants to reach $\partial\Omega$ and to do it in the shortest time as possible.

Two important questions arise:

(A) Can always Player I reach $\partial\Omega$?

(B) Which is the optimal strategy for Player I?
In other words, which is, at any step, the choice of $\boldsymbol{v}$ that leads Player I to reach $\partial\Omega$ as fast as possible?

The answer to question (A) is "yes if the set Ω is convex".
To give instead an answer to question (B), we are first going to study the particular case of a ball, i.e. $\Omega = B_R(p)$, for some $R > 0$ and $p \in \mathbb{R}^2$.
As usual, without loss of generality, we may assume $p = 0$.

Let us fix $\varepsilon > 0$ and build a first crown Ω_1, from where Player I can win in just one step. Hence we consider the set

$$C_1 := \left\{ z \in \Omega \,\Big|\, z = \frac{y_1 + y_2}{2}, \text{ with } y_1, y_2 \in \partial\Omega \text{ and } |y_2 - y_1| = 2\sqrt{2}\,\varepsilon \right\}.$$

By Pythagoras' Theorem, C_1 is the circle centered at $p = 0$ with radius $R_1 = \sqrt{R^2 - 2\varepsilon^2}$. This means that the distance between C_1 and $\partial\Omega$ is equal to $d_1 := R - \sqrt{R^2 - 2\varepsilon^2}$.

The crown Ω_1 is the open portion of plain between $\partial\Omega$ and C_1, i.e.

$$\Omega_1 = \{z \in \mathbb{R}^2 \,|\, 0 < d(z, \partial\Omega) < d_1\},$$

see Figure 1.
Note that, if $\varepsilon > 0$ small enough, then $d_1 \approx \frac{\varepsilon^2}{R}$ (show this as an exercise).

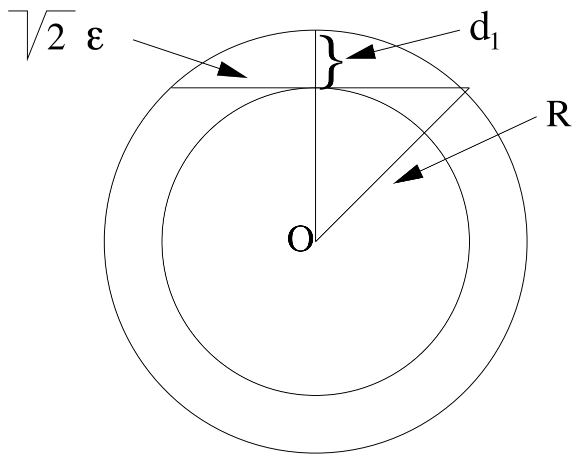


Figure 6: `The first crown` $\Omega_1 \subset \Omega = B_R(0)$.

Let us assume that $x \in C_1$, then Player I can always reach $\partial\Omega$ in exactly one step. In fact, if Player I chooses one of the two tangent directions $\boldsymbol{v}_1$ or $\boldsymbol{v}_2$, whatever the choice of Player II is, the game evolves into a point y_1 or y_2 of the boundary, as one can see in Figure 2.
Therefore, if Player I "plays good", then in just one step he can win.

The previous remark still holds if $x \in \Omega_1$. In fact, Player I can still chooses one of the two directions tangent to the crown passing through the point x and he can cross $\partial\Omega$ in only one step.

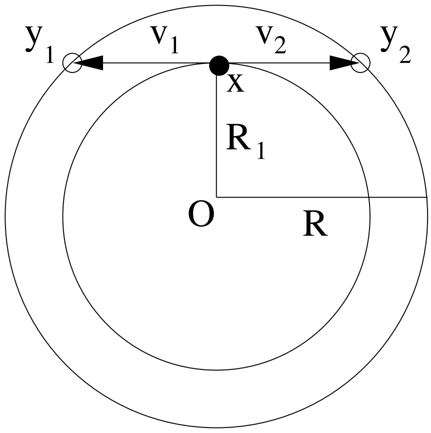


Figure 7: `The two possible evolutions of the game from the point` x`, following an optimal choice for Player I.`

Similarly, one can build a second crown, looking at

$$C_2 := \left\{ z \in \Omega \,\middle|\, z = \frac{y_1 + y_2}{2}, \text{ with } y_1, y_2 \in \partial\Omega \text{ and } |y_2 - y_1| = 4\sqrt{2}\varepsilon \right\}.$$

Then the second crown Ω_2, is the open portion of plain between C_2 and C_1. If Player I moves always in a direction tangent to the crown passing from the point itself, in two steps he will reach $\partial\Omega$ and he will win.
Analogously the kth crown is the set from where Player I can reach $\partial\Omega$ is exactly $k \leq 1$ steps and it is given by the portion of plan between C_{k-1} and C_k, where

$$C_k := \left\{ z \in \Omega \,\middle|\, z = \frac{y_1 + y_2}{2}, \text{ with } y_1, y_2 \in \partial\Omega \text{ and } |y_2 - y_1| = 2k\sqrt{2}\varepsilon \right\}.$$

Therefore an optimal strategy for Player I consists in moving always in a direction tangent to the crown passing from that point.

It is clear that optimal strategies are unique. In fact at any point, Player I can chooses a tangent direction or the opposite one (which is still tangent).

The numbers of steps necessary to reach $\partial\Omega$ following an optimal strategy is easy to estimate. In fact, we have already remarked that, whenever $\varepsilon > 0$ small enough, the radius of each crown is $d \approx \frac{\varepsilon^2}{R}$.
Therefore Player I needs approximately n steps where n is given by

$$n = \frac{R^2}{\varepsilon^2}.$$

The natural number n is called *exit time.*

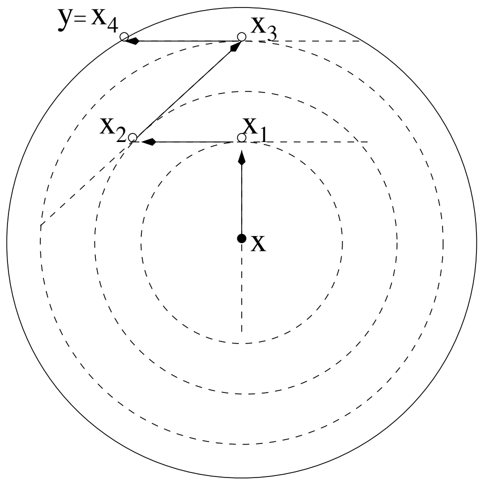


Figure 8: An optimal configuration of the game starting from the origin $x = 0$. The game ends at the point y in exactly 4 steps.

We may also notice that Player II cannot neither stop nor slow down Player I, whenever Player I chooses an optimal direction.

Now let us investigate what happens if the set Ω is not a ball.
If the set is convex then it is always possible to build crowns exactly as in the case of the ball and what we have shown in the particular case $\Omega = B_R(0)$ is still correct. In Figures 4 and 5, you can get an idea of the crowns when Ω is a square or a triangle; as an exercise, build the crowns for ellipses.

Instead whenever the set is not convex the crowns do not anymore entirely belong to the set Ω and when the game is near the non-convex part of the set, Player II can always push the game far away from the boundary.

Therefore we consider now just convex sets.

In this case it is possible to show that *the exit times of the previous games give, as $\varepsilon \to 0^+$, an approximation of the generalized evolution by mean curvature flow.*

Definition 7.7. Given a game as above with step $\varepsilon > 0$, we indicate by $U^\varepsilon(x)$ the function given by the *minimum exit time* from the point $x \in \Omega$, i.e.

$$U^\varepsilon(x) = \varepsilon^2 k,$$

where k is the numbers of steps that Player I needs to reach $\partial\Omega$ starting from the point $x \in \Omega$ and following an optimal strategy.

We quote the following two results without proofs.

Proposition 7.1 (Dynamic Programming Principle)**.** Let be $\varepsilon > 0$ and $U^\varepsilon : \Omega \to \mathbb{R}$ be the exit-time given in Definition 7.7, then for any $x \in \Omega$

$$U^\varepsilon(x) = \min_{\|\boldsymbol{v}\|=1} \max_{b=\pm 1} \left\{\varepsilon^2 + U^\varepsilon(x + \sqrt{2}\varepsilon b\boldsymbol{v})\right\}.$$

Remark 7.5. As we will see later, in the theory of differential games, the Dynamic Programming Principle assumes always the form of a max-min or of a min-max. This is one of the main differences with the classic control theory and expresses the fact that, for Player I, it is not enough to minimize the pay-off but he has to minimize it, thinking that Player II is playing as good as possible. The fact that the Dynamic Programming Principle is a min-max instead of a max-min means that we are looking at the game from the point of view of Player I. All these remarks will be clarified in the next section.

Exactly as for the value function of a control problem, the previous Dynamic Programming Principle is the key-step in order to associate a PDE to the exit-time of a differential game.

Theorem 7.1. Let be $\varepsilon > 0$ and $U^\varepsilon : \Omega \to \mathbb{R}$ be the exit-time given in Definition 7.7, then (up to a subsequence) $U^\varepsilon(x)$ converges uniformly in Ω to a function $U(x)$. Moreover the limit-function $U(x)$ is the unique viscosity solution of the Dirichlet problem:

$$\begin{cases} \Delta\, U(x) - \Delta_\infty\, U(x) + 1 = 0, & x \in \Omega, \\ U(x) = 0, & x \in \partial\Omega. \end{cases}$$

Note that the PDE given in the previous theorem is "mean curvature= 1", but we are not interested in function related to a constant mean curvature but to the corresponding evolution problem.

So to conclude this section we are going to show briefly how it is possible to go from a game associated to a stationary equation (i.e. to a Dirichlet problem) to a game associated to an evolution equation (i.e. to a Cauchy problem).

Therefore, let us now consider an *objective function* u_0, which expresses the function we want to optimize and let $T > 0$ be a fixed time when the game ends, which is in general called *maturity time*.

The goal for Player I is not anymore to go out from a fixed bounded domain but to minimize the objective function when the game reaches the maturity time. This means that we need to consider the *value function*:

$$u^\varepsilon(t,x) = \min_{y_{t,x}(s)} u_0\left(y_{t,x}(T)\right), \tag{93}$$

where $y_{t,x}(s)$ are all the possible piecewise linear paths which give the history of the game, starting at the time t (with $0 < t < T$) from a point $x \in \mathbb{R}^2$.

As we have done for exit-time $U^\varepsilon(x)$, using a suitable Dynamic Programming Principle, as $\varepsilon \to 0^+$, it is possible to associate a PDE to the limit-function of $u^\varepsilon(t,x)$.

Proposition 7.2 (Dynamic Programming Principle)**.** Let be $\varepsilon > 0$ and $u^\varepsilon : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ be the value function (93), then for any $t \in [0,T]$ and for any $x \in \mathbb{R}^2$

$$u^\varepsilon(t,x) = \min_{\|\boldsymbol{v}\|=1} \max_{b=\pm 1} u^\varepsilon(t+\varepsilon^2, x + \sqrt{2}\varepsilon b \boldsymbol{v}).$$

Theorem 7.2. Let be $\varepsilon > 0$ and $u^\varepsilon : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ be the value function (93), then (up to a subsequence) $u^\varepsilon(t,x)$ converges uniformly in $[0,T] \times \mathbb{R}^n$ to a function $u(t,x)$. Moreover the limit-function $u(t,x)$ is the unique viscosity solution of the following problem:

$$\begin{cases} u_t + \Delta\, u - \Delta_\infty\, u = 0, & \text{in } [0,T) \times \mathbb{R}^n, \\ u = u_0, & \text{in } \{T\} \times \mathbb{R}^n. \end{cases} \tag{94}$$

The previous theorem tells that, the Kohn-Serfaty's game gives a discretization with step $\varepsilon > 0$ of the viscosity solution for the backward level-set equation for the evolution by mean curvature flow.
To get an approximation of the generalized evolution by mean curvature flow for the initial hypersurface $\Sigma_0 := \{x \in \mathbb{R}^2 \mid u_0(x) = 0\}$, it is sufficient to reverse the time, which means to look at the function

$$v(t) := u(T-t), \quad \forall\, t \geq 0.$$

If $u(t,x)$ is the viscosity solution of problem (94), then $v(t,x)$ is the viscosity solution of the Cauchy problem:

$$\begin{cases} v_t - \Delta\, v + \Delta_\infty\, v = 0, & \text{in } (0,+\infty) \times \mathbb{R}^n, \\ v = u_0, & \text{in } \{0\} \times \mathbb{R}^n. \end{cases} \tag{95}$$

A similar game can be introduced to study surfaces evolving in $\mathbb{R}^3$.
This game is very similar to the one we have presented but Player I has to choose two vectors $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^3$ such that $\|\boldsymbol{v}\| = \|\boldsymbol{w}\| = 1$ and $\boldsymbol{v} \perp \boldsymbol{w}$, while Player II chooses two numbers $b, \beta \in \{-1, +1\}$ and the game evolves from a point x to a point $y := x + \sqrt{2}\,\varepsilon(b\,\boldsymbol{v} + \beta\,\boldsymbol{w})$.
To generalize the game in $\Omega \subset \mathbb{R}^n$ with $n \geq 4$ is given as an open question.

The previous game works just when the initial evolving hypersurface is convex. Whenever the Σ_0 is not convex, the game gives as limit the evolution by positive mean curvature flow since the non-convex part of Σ_0 does not move. The evolution by positive mean curvature flow is much less interesting evolution to study because it is not anymore associated to the gradient-flow of the area.

To find a generalization of this game which can give, as limit, the generalized evolution by mean curvature flow also for non-convex sets, it is a very interesting open problem.

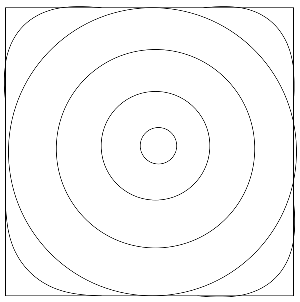


Figure 9: `Behavior of the crowns for the square.`

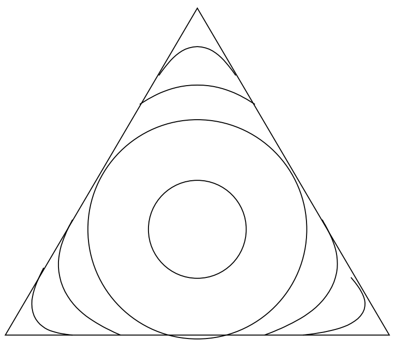


Figure 10: `Behavior of the crowns for the triangle.`

8 Differential games.

In this section we give a brief overview on *two-person zero-sum differential games.* For more details we refer to the book of Bardi and Capuzzo Dolcetta, Chapter VIII.

Roughly speaking, to study a two-person zero-sum differential game means to consider optimal control problems depending on two different families of controls, giving respectively the set of choices for the player I and the set of choices for the player II and a cost functional that the player I tries to minimize while the player II tries to maximize. Therefore we can consider that the cost for player I is minus the cost for player II. This explains the term "zero-sum" which means that the total cost (i.e. the sum of the cost for player I and the cost for player II) is always zero.

Therefore, let us introduce two compact metric spaces A and B: usually one can think of two compact subsets of $\mathbb{R}^m$. E.g in the Kohn-Serfaty's game $A = \partial B_1(0)$ (or also $A = \overline{B_1(0)}$) while $B = \{-1, +1\}$.

Let us define the two following sets of controls:

$$\mathcal{A} := \{\alpha : [0, +\infty) \to A \text{ measurable}\},$$
$$\mathcal{B} := \{\beta : [0, +\infty) \to B \text{ measurable}\},$$

which give respectively the set of controls for the player I and the set of controls for player II.

We look at continuous *dynamics* $f : \mathbb{R}^n \times A \times B \to \mathbb{R}^n$. Then the corresponding control systems are

$$\begin{cases} \dot{y}(t) = f(y(t), \alpha(t), \beta(t)), & t > 0, \\ y(0) = x. \end{cases} \tag{96}$$

We indicate by $y_x^{\alpha,\beta}(t)$ the solution of (96), which we recall to be given by

$$y_x^{\alpha,\beta}(t) = x + \int_0^t f(y_x^{\alpha,\beta}(s), \alpha(s), \beta(s))ds.$$

The goal of the player I is to minimize $J(x; \alpha, \beta)$ while the goal of player II is to maximize $J(x; \alpha, \beta)$.
In other words, J is the cost that player I has to pay while $-J$ is the cost that player II has to pay (which implies, as we have already remarked, that the total cost is zero).

In addition we assume the following condition for the dynamics, i.e.

$$\big(f(x,a,b) - f(y,a,b)\big) \cdot (x-y) \le L|x-y|^2, \tag{97}$$

for some $L > 0$ and for all $x, y \in \mathbb{R}^n$ and for any $a \in A$ and $b \in B$.

Under all the previous assumptions, the following properties for the solutions of (96) hold:

(i) $|y_x^{\alpha,\beta}(t) - y_z^{\alpha,\beta}(t)| \le \mathrm{e}^{Lt}|x-z|$, for all $x, z \in \mathbb{R}^n$ and $t > 0$,

(ii) $|y_x^{\alpha,\beta}(t) - x| \le M_x t$, for any $t \in [0, 1/M_x]$,

(iii) $|y_x^{\alpha,\beta}(t)| \le (|x| + \sqrt{2Kt})\,\mathrm{e}^{Kt}$, for all $x \in \mathbb{R}^n$ and $t > 0$,

where the constant L is given by assumption (97), while

$$M_x := \max\{|f(z,a,b)| \,|\, |x-z| \le 1, a \in A, b \in B\},$$

and

$$K := L + \max\{|f(0,a,b)| \,|\, a \in A, b \in B\}.$$

Now we introduce the notion of strategies which characterize the fact that, in the theory of differential games, usually the choices of the a player depend on the choices of the other player.

Definition 8.1. A *strategy* for the player I is a map $\gamma : \mathcal{B} \to \mathcal{A}$ while a strategy for the player II is a map $\gamma : \mathcal{A} \to \mathcal{B}$.

A strategy for the player I is called *non-anticipating* if, for any $t > 0$,

$$\beta(s) = \widetilde{\beta}(s), \quad \forall\, s \le t \;\Rightarrow\; \gamma[\beta](s) = \gamma[\widetilde{\beta}](s), \quad \forall\, s \le t.$$

The same definition holds for the strategies of player II.

We indicate respectively by Γ and Δ, the set of all the non-anticipating strategies for the player I and the set of all the non-anticipating strategies for the player II.

Example 8.1. In the Kohn-Serfaty's game the strategies are non-anticipating while in a game like chess one in general has to consider that the strategies are anticipating, since the choice of each player depends also on the future choice of the other player.

As we did for the optimal control problems, now we want to introduce the "value" of the game associated to some functional $J : \mathbb{R}^n \times \mathcal{A} \times \mathcal{B} \to \mathbb{R}$.
In the theory of differential games, in general, there are two different values, depending if you look at the game from the point of view of the player I or from the point of view of the player II.

Definition 8.2 (Upper and lower values)**.** Given a cost-functional $J(x; \alpha, \beta)$, the *lower value* of the game is defined as

$$v(x) := \inf_{\gamma \in \Gamma} \sup_{\beta \in \mathcal{B}} J(x; \gamma[\beta], \beta);$$

while the *upper value* of the game is defined as

$$u(x) := \sup_{\delta \in \Delta} \inf_{\alpha \in \mathcal{A}} J(x; \alpha, \delta[\alpha]).$$

If $v(x) = u(x)$, we say that the game with initial point x has a value.

Remark 8.1. Note that

$$u(x) \leq v(x), \quad \text{for all } x \in \mathbb{R}^n.$$

To prove the inequality is not easy. Heuristically, at each instant of time the first player knows the choice that the second one is making at the same time.

A limit case of the described game is the *static game* (which is easier to study but much less interesting for the applications).
A static game is a game where the player I chooses on the set $\mathcal{A}$ and the player II on the set $\mathcal{B}$ without any interaction between the two players.

In the case of a static game the lower and upper values are simply given by:

$$\begin{aligned} v_s(x) &= \sup_{\beta \in \mathcal{B}} \inf_{\alpha \in \mathcal{A}} J(x; \alpha, \beta), \\ u_s(x) &= \inf_{\alpha \in \mathcal{A}} \sup_{\beta \in \mathcal{B}} J(x; \alpha, \beta). \end{aligned}$$

In this case, it is trivial to note that $v_S(x) \leq u_S(x)$, for any $x \in \mathbb{R}^n$. In fact, for any $\alpha \in \mathcal{A}$ and $\beta \in \mathcal{B}$,

$$\inf_{\alpha \in \mathcal{A}} J(x; \alpha, \beta) \leq J(x; \alpha, \beta) \leq \sup_{\beta \in \mathcal{B}} J(x; \alpha, \beta).$$

More in general, one could prove:

$$v_S(x) \leq v(x) \leq u(x) \leq u_S(x).$$

Therefore we can note that is much easier that an "interacting game" has a value than the corresponding static game has.

While we were studying the optimal control problem, we have looked mainly at the case of time-depending functionals, which is in general associated to a Cauchy problem for an evolution Hamilton-Jacobi-Bellman equation. This case is in general called *finite horizon problem*.

In this section we will instead pay more attention to the so called *infinite horizon problem*, which is associated to a functional time-independent and it is related to a stationary equation.

Let us fix $\lambda > 0$, and look at a continuous *running cost* $l : \mathbb{R}^n \times A \times B \to \mathbb{R}$. We also assume that the running cost is Lipschitz and bounded in x, uniformly in $A \times B$.

Then the functional can be written as

$$J(x; \gamma[\beta], \beta) = \int_0^{+\infty} l\big(y_x^{\gamma[\beta],\beta}(t), \gamma[\beta](t), \beta(t)\big) \mathrm{e}^{-\lambda t} dt,$$

and analogously,

$$J(x; \alpha, \delta[\alpha]) = \int_0^{+\infty} l\big(y_x^{\alpha,\delta[\alpha]}(t), \alpha(t), \delta[\alpha](t)\big) \mathrm{e}^{-\lambda t} dt.$$

We state the following regularity result for the lower and upper values of the game. A proof can be found in the book of Bardi and Capuzzo Dolcetta (Proposition VIII.1.8.)

Proposition 8.1. Under all the previous assumptions, the lower and the upper values $u(x)$ and $v(x)$ are bounded and uniformly continuous in $\mathbb{R}^n$. Moreover they are Hölder continuous with exponent γ, with

$$\gamma = \begin{cases} 1, & \text{if } \lambda > L, \\ \text{any } \gamma < 1, & \text{if } \lambda = L, \\ \dfrac{\lambda}{L}, & \text{if } \lambda < L. \end{cases}$$

where $L > 0$ is the constant given in assumption (97).

Remark 8.2. We recall that for the optimal control problem we have showed that the value is Lipschitz (see Remark 3.3).

We are interested in associating a PDE to the values of the game.

The main point is again to show that a Dynamic Programming Principle. For a proof see the book of Bardi and Capuzzo Dolcetta (Theorem VIII.1.9).

Proposition 8.2 (Dynamic Programming Principle)**.** Under all the previous assumptions, then for all $x \in \mathbb{R}^n$ and for any $t > 0$, it holds

$$v(x) = \inf_{\gamma\in\Gamma} \sup_{\beta\in\mathcal{B}} \left\{ \int_0^{+\infty} l\big(y_x^{\gamma[\beta],\beta}(s), \gamma[\beta](s), \beta(s)\big) \mathrm{e}^{-\lambda s} ds + v(y_x^{\gamma[\beta],\beta}(t)) \mathrm{e}^{-\lambda t} \right\},$$

$$u(x) = \sup_{\delta\in\Delta} \inf_{\alpha\in\mathcal{A}} \left\{ \int_0^{+\infty} l\big(y_x^{\alpha,\delta[\alpha]}(s), \alpha(s), \delta[\alpha](s)\big) \mathrm{e}^{-\lambda s} ds + u(y_x^{\alpha,\delta[\alpha]}(t)) \mathrm{e}^{-\lambda t} \right\}.$$

Using the Dynamic Programming Principle, it is possible to show that the lower and the upper values solve, respectively, two (a priori) different Hamilton-Jacobi-Bellman equations. More precisely:

Theorem 8.1. Under all the previous assumptions, the lower value $v(x)$ is a viscosity solution of the following stationary Hamilton-Jacobi-Bellman equation:

$$\lambda v + H_1(x, Dv) = 0, \quad \text{in } x \in \mathbb{R}^n,$$

where

$$H_1(x,p) := \min_{b\in B} \max_{a\in A} \big\{ -f(x,a,b)\cdot p - l(x,a,b) \big\}.$$

The upper value $u(x)$ is a viscosity solution of the following stationary Hamilton-Jacobi-Bellman equation:

$$\lambda u + H_2(x, Du) = 0, \quad \text{in } x \in \mathbb{R}^n,$$

where

$$H_2(x,p) := \max_{a\in A} \min_{b\in B} \big\{ -f(x,a,b)\cdot p - l(x,a,b) \big\}.$$

The proof is very similar to the one that we gave for the optimal control problem. One can find a proof in the book of Bardi and Capuzzo Dolcetta (Theorem VIII.1.10).

Note that in general $H_1(x,p) \neq H_2(x,p)$.

Nevertheless, whenever $u(x) = v(x)$ (which means that the game has a value starting from any point $x \in \mathbb{R}^n$), then $H_1(x,p) = H_2(x,p)$.

9 Exercises.

Exercise 9.1. Let be

$$u(x) = \begin{cases} x, & 0 < x \le \dfrac{1}{2}, \\ 1 - x, & \dfrac{1}{2} < x < 1. \end{cases}$$

(i) Show that $u(x)$ is a viscosity solution of $|u'(x)| - 1 = 0$ on $(0, 1)$.

(ii) Show that $u(x)$ is not a viscosity solution of $-|u'(x)| + 1 = 0$ on $(0, 1)$.

(iii) Find a viscosity solution of $-|u'(x)| + 1 = 0$ on $(0, 1)$.

Exercise 9.2. Given the Cauchy problem:

$$\begin{cases} u_t + |u_x|^2 = 0, & (0, +\infty) \times \mathbb{R}, \\ u = 0, & \{0\} \times \mathbb{R}, \end{cases}$$

(i) show that $u_1(t, x) = 0$ is a viscosity solution.

(ii) show that

$$u_2(t, x) = \begin{cases} 0, & |x| \ge t \\ x - t, & 0 \le x < t \\ -x - t, & -t < x \le 0 \end{cases}$$

is not a viscosity solution.
Hint: show that u_2 is not a viscosity supersolution at points $(t_0, x_0) = (\bar{t}, 0)$ with $t_0 > 0$, using as test-function $\varphi(t, x) = -t$.

Exercise 9.3. Show by exhibiting an example that is false in general that, if u_1 and u_2 are viscosity solutions of $H(Du) = 0$, the same is true for $u_1 \wedge u_2$ and $u_1 \vee u_2$.
Hint: look at $|x| - 1$ and $-|x| + 1$, respectively, as maximum and minimum of two lines and use what you know about the eikonal equation.

Exercise 9.4. Using the Perron's method and assuming the validity of comparison principles, prove the existence of a viscosity solution for the Dirichlet problem:

$$\begin{cases} |u'(x)| - f(x) = 0, & x \in (0, 1), \\ u(0) = u(1) = 0, \end{cases}$$

where f is a continuous on $[0, 1]$ and $f \ge 0$.
Solution: consider the viscosity subsolution $\underline{u}(x) = 0$ and the viscosity supersolution $\overline{u}(x) = -M\left|x - \frac{1}{2}\right| + \frac{M}{2}$, where $M := \max_{x \in [0,1]} f(x)$.

Exercise 9.5. Let $u : \mathbb{R} \to \mathbb{R}$ be a viscosity solution of

$$F(x, u, u', u'') = 0.$$

Write the equation solved by $v(x) = \mathrm{e}^{\frac{u(x)}{\varepsilon}}$ and show that v is indeed a viscosity solution of the found equation.
Solution: v is a viscosity solution of $F\left(x, \varepsilon \ln(v), \frac{\varepsilon}{v}\, v', -\frac{\varepsilon}{v^2}\,(v')^2 + \frac{\varepsilon}{v}\, v''\right) = 0$.

Exercise 9.6. Let $u : \mathbb{R} \to \mathbb{R}$ be a viscosity solution of

$$F(x, u, u') = 0.$$

Write the equation solved by $v(y) = u \circ \Phi(y)$, where $\Phi(y) = \mathrm{e}^{\frac{x}{\varepsilon}}$ and show that v is indeed a viscosity solution of the equation you found.
Solution: let $y = \Phi^{-1}(x) = \varepsilon \ln x$,
then $v(y)$ is a viscosity solution of $F\left(y, v(\varepsilon \ln y), \frac{\varepsilon}{y}\; Dv(\varepsilon\, \ln y)\right) = 0$.

Exercise 9.7. Show that, given $u : [0, T] \to \mathbb{R}$ continuous, the following properties are equivalent:

(i) u is non-decreasing in $[0, T]$,

(ii) $u'(x) \geq 0$, in the viscosity sense, in $[0, T]$.

Hint: to show that $(ii) \Rightarrow (i)$, use the following claim to get a contradiction. Claim: given 3 points $0 < t_1 < \bar{t} < t_2 < T$ such that $u(t_1) > u(\bar{t}) > u(t_2)$, then there exists $\varphi \in C^1([t_1, t_2])$ such that

$$\begin{cases} \varphi(t_1) = u(t_1), \\ \varphi(t_2) = u(t_2), \\ \varphi(\bar{t}) < u(\bar{t}), \\ \varphi'(t) < 0, \quad t \in (t_1, t_2). \end{cases}$$

Exercise 9.8. Using Exercise 9.7, show that, given $u : [0, T] \to \mathbb{R}$ continuous, then

$$u'(x) = 0, \text{ in the viscosity sense, in } [0, T] \iff u = \text{constant in } [0, T].$$

Exercise 9.9. Given a 1-dimensional control problem, with $n = 1$, $A = \{1, -1\}$, $f(x, a) = a$ and

$$J(x; \alpha) = \int_0^{+\infty} \mathrm{e}^{\lambda t}\; l(y_x^\alpha(t))dt,$$

where $\lambda > 0$ and the function $l : \mathbb{R} \to \mathbb{R}$ smooth, and such that

$$\begin{cases} l(x) = l(-x), \\ \max\limits_{\mathbb{R}} l(x) = l(0) > 0, \\ l(x) = 0, \qquad \text{for } |x| > R, \\ x\, l'(x) < 0, \quad \text{for } |x| < R, \end{cases}$$

for some $R > 0$.
Find the explicit expression for the value function.
Hint: observe that the optimal control α^* is constant in time, then, for any x, $\alpha^*(t) = 1$ or $\alpha^*(t) = -1$ and use the properties of the running cost $l(x)$ to conclude.

Exercise 9.10. Under the assumption (46) and (57), show that the Hopf-Lax formula (58) is globally Lipschitz in $\mathbb{R}^n \times [0, +\infty)$.
Hint: Use that the infimum in (58) is attained in some point to get the Lipschitz continuity in space and then the Functional Identity (59) in order to get the Lipschitz property in time (see the proof of Lemma 4.1).

Exercise 9.11. Let be $H(p) = \frac{|p|^2}{2}$ and $g(x) = \frac{|x|^2}{2}$.
Find the viscosity solution of (45).
Solution: $u(t, x) = \frac{|x|^2}{2(t+1)}$.

Exercise 9.12. Let be $H(p) = \frac{|p|^2}{2}$ and $g(x) = -|x|$.
Find the viscosity solution of (45).
Solution: $u(t, x) = -|x| - \frac{t}{2}$.

Exercise 9.13. Write the solution of (45) with $H(p) = \frac{|p|^\alpha}{\alpha}$ with $\alpha > 1$, for any g Lipschitz continuous initial datum.

Exercise 9.14. Given $H(p) = \frac{|p|^\alpha}{\alpha}$ with $\alpha > 1$ and g bounded, show that the infimum in (58) is a minimum and it is attained in the ball centered in the origin and with radius $R(t) = \big(2\beta\big)^{\frac{1}{\beta}} t^{\frac{\beta-1}{\beta}} \|g\|_\infty^{\frac{1}{\beta}}$, where $\beta = \frac{\alpha}{\alpha-1} > 1$.
Hint: Note that the infimum in (58) is smaller or equal to $\|g\|_\infty$.
Then, using the explicit formula found in the Exercise 9.13, show that, for any $y \in \mathbb{R}^n \backslash \overline{B_R(t)}(x)$, $f(y) > \|g\|_\infty$, where $f(y) = g(y) + \frac{|x-y|^\beta}{t^{\beta-1}}$ and $\beta = \frac{\alpha}{\alpha-1}$.

Exercise 9.15. Let be $H(p) = |p|$, write the Hopf-Lax formula (58);
Solution: $u(t, x) = \inf\{g(y) \,|\, |x - y| \leq t\}$.
Note that the minimum is attained in the ball $\overline{B_R(t)}(x)$ with $R(t) = t$.

Exercise 9.16. Let us look at the Cauchy problem:

$$\begin{cases} u_t(t,x) + b \cdot Du = 0, & x \in \mathbb{R}^n, t > 0, \\ u(0,x) = g(x), & x \in \mathbb{R}^n. \end{cases} \tag{98}$$

and assuming g Lipschitz.

(i) Show that the Hopf-Lax formula (58) is given by

$$u(t,x) = g(x - tb).$$

Hint: show that if $H(p) = b \cdot p$ then the Legendre-Fenchel transform is given by $H^*(p) = +\infty$ for any $p \neq b$ and $H(b) = 0$.

(ii) Is the condition (46) satisfied?

Exercise 9.17. Let be $\delta > 0$ and $g^\delta(x)$ the sup-convolution of $g(x)$, i.e.

$$g^\delta(x) = \sup_{y \in \mathbb{R}^n} \left[g(y) - \frac{|x-y|^2}{2\delta} \right]. \tag{99}$$

Assuming that g is bounded on $\mathbb{R}^n$, show that $\left\| g^\delta \right\|_\infty \leq \|g\|_\infty$.

Exercise 9.18. Assuming that g is bounded on $\mathbb{R}^n$ and $g^\delta(x)$ is the sup-convolution given by (99), prove that the supremum is attained in the ball $\overline{B_R(\delta)}(x)$ with $R(\delta) = 2\sqrt{\delta \; \|g\|_\infty}$.

Exercise 9.19. Assuming that g is bounded and Lipschitz on $\mathbb{R}^n$ and $g^\delta(x)$ is the sup-convolution given by (99), prove that, for any δ, $g^\delta(x)$ is locally Lipschitz continuous in $\mathbb{R}^n$.
Hint: Proceed similarly to Exercise 9.10.

Exercise 9.20. Assuming that g is bounded on $\mathbb{R}^n$ and $g^\delta(x)$ is the sup-convolution given by (99), prove that, for any δ, $g^\delta(x)$ is locally Lipschitz continuous in $\mathbb{R}^n$.
Hint: Proceed similarly to Lemma 4.1.

Exercise 9.21. Let be $g(x) = |x|$ and $\delta > \varepsilon > 0$, calculate $u(x) := \big(g_\delta(x)\big)^\varepsilon$, where $g_\delta(x)$ is the inf-convolution given by (72) while $g^\varepsilon(x)$ is the sup-convolution given by (73). Check that $u \in C^{1,1}(\mathbb{R}^n)$ but not C^2.
Solution:

-A. First step: Calculate $g_\delta(x) = \inf_{y \in \mathbb{R}^n} f(y)$ where $f(y) = |y| + \frac{|x-y|^2}{2\delta}$.

A.1: Check the limit as $|y| \to +\infty$.

A.2: Note that the unique non-differentiability point of $f(y)$ is $y_0 = 0$.

A.3: Note that the stationary point is given by $\overline{y} = \frac{|x|-\delta}{|x|} x$ and it is admissible if and only if $|x| \geq \delta$.

A.4: Compare $f(y_0)$ and $f(\overline{y})$ in the case when $|x| \geq \delta$.

A.5: Conclude that

$$g_\delta(x) = \begin{cases} \dfrac{|x|^2}{2\delta}, & |x| < \delta, \\ |x| - \dfrac{\delta}{2}, & |x| \geq \delta. \end{cases}$$

-B. Second step: Calculate the sup-convolution of the previous function $g_\delta(x)$ assuming $\delta > \varepsilon > 0$.

B.1: Observe that

$$\big(g_\delta(x)\big)^\varepsilon = \sup_{y\in\mathbb{R}^n} \left\{ \sup_{|y|\leq\delta} \left[\frac{|y|^2}{2\delta} - \frac{|x-y|^2}{2\varepsilon}\right]; \sup_{|y|\geq\delta} \left[|y| - \frac{\delta}{2} - \frac{|x-y|^2}{2\varepsilon}\right] \right\} =: \sup \big\{I_1(x); I_2(x)\big\}.$$

B.2: Note that $f(y) = \frac{|y|^2}{2\delta} - \frac{|x-y|^2}{2\varepsilon}$ is a parabola, therefore the supremum is attained in the stationary point $\overline{y}$, whenever $|\overline{y}| \leq \delta$ otherwise the supremum is attained on the boundary $|y| = \delta$.

B.3: Calculate f at the stationary point which is $\overline{y} = \frac{\delta}{\delta-\varepsilon} x$ and note that the point is admissible whenever $|x| \leq \delta - \varepsilon$. This means that

$$I_1(x) = \begin{cases} \dfrac{|x|^2}{2(\delta-\varepsilon)}, & |x| \leq \delta - \varepsilon, \\ \displaystyle\sup_{|y|=\delta} \left[\frac{\delta}{2} - \frac{|x-y|^2}{2\varepsilon}\right], & |x| > \delta - \varepsilon. \end{cases}$$

B.4: To calculate $I_2(x)$, one has to proceed similarly.
First note that, since $|y| \geq \delta > \varepsilon > 0$, then $h(y) = |y| - \frac{|x-y|^2}{2\varepsilon} - \frac{\delta}{2}$ is differentiable at any point of the domain.

B.5: Calculate the stationary point which is $\overline{y} = \frac{|x|+\varepsilon}{|x|} x$ which is admissible if and only if $|x| \geq \delta - \varepsilon$.

B.6: Therefore, you get

$$I_2(x) = \begin{cases} |x| + \dfrac{\varepsilon - \delta}{2}, & |x| \geq \delta - \varepsilon, \\ \sup\limits_{|y|=\delta} \left[\dfrac{\delta}{2} - \dfrac{|x-y|^2}{2\varepsilon} \right], & |x| < \delta - \varepsilon. \end{cases}$$

B.7: Using that $f(y) = h(y)$ whenever $|y| = \delta$, it is easy to show that $I_1(x) \geq I_2(x)$ when $|x| \leq \delta - \varepsilon$ while $I_1(x) \leq I_2(x)$ when $|x| \geq \delta - \varepsilon$. Therefore, one can conclude

$$u(x) = \big(g_\delta(x)\big)^\varepsilon = \begin{cases} |x| + \dfrac{\varepsilon - \delta}{2}, & |x| \geq \delta - \varepsilon, \\ \dfrac{|x|^2}{2(\delta - \varepsilon)}, & |x| < \delta - \varepsilon. \end{cases}$$

C. Third step: Check that the previous function is $C^{1,1}(\mathbb{R}^n)$ but not C^2. In fact

$$Du(x) = \begin{cases} \dfrac{x}{|x|}, & |x| > \delta - \varepsilon, \\ \dfrac{x}{\delta - \varepsilon}, & |x| < \delta - \varepsilon, \end{cases}$$

which is continuous and Lipschitz at any point while

$$D^2u(x) = \begin{cases} \dfrac{1}{|x|} \left(Id - \dfrac{x}{|x|} \otimes \dfrac{x}{|x|} \right), & |x| > \delta - \varepsilon, \\ \dfrac{1}{\delta - \varepsilon} Id, & |x| < \delta - \varepsilon, \end{cases}$$

which is not continuous at the points $|x| = \delta - \varepsilon$.

Exercise 9.22. Do as in Exercise 9.21, starting from the function $g(x) = -|x|$.

Exercise 9.23. What happens in the Exercises 9.21 and 9.22, assuming $0 < \delta < \varepsilon$? Is $\big(g_\delta(x)\big)^\varepsilon \in C^{1,1}$?
and what can you say about the limit case $0 < \delta = \varepsilon$?

10 Appendix: Semicontinuity.

We recall the definitions of liminf and limsup at a point $x_0 \in \mathbb{R}^n$:

$$\liminf_{x \to x_0} f(x) := \sup_{r>0} \inf_{x \in B_r(x_0) \setminus \{x_0\}} f(x),$$
$$\limsup_{x \to x_0} f(x) := \inf_{r>0} \sup_{x \in B_r(x_0) \setminus \{x_0\}} f(x).$$

Remark 10.1. Note that, if we set

$$f_r(x_0) = \inf_{x \in B_r(x_0) \setminus \{x_0\}} f(x),$$
$$f^r(x_0) = \sup_{x \in B_r(x_0) \setminus \{x_0\}} f(x),$$

then $f_r(x_0)$ is non-increasing in $r > 0$ while $f^r(x_0)$ is non-decreasing.

Therefore the liminf and limsup are both well-defined and are the limits as $r \to 0^+$ of $f_r(x_0)$ and $f^r(x_0)$, respectively.

Remark 10.2. Recall that

$$\liminf_{x \to x_0} f(x) \leq \limsup_{x \to x_0} f(x),$$

for any $x_0 \in \mathbb{R}^n$ (easy to check).

Definition 10.1. Given $f : \Omega \to \mathbb{R}$ with $\Omega \subset \mathbb{R}^n$ open and $x_0 \in \Omega$, then

1. f is *lower semicontinuous* at x_0 if

$$\liminf_{x \to x_0} f(x) \geq f(x_0);$$

2. f is *upper semicontinuous* at x_0 if

$$\limsup_{x \to x_0} f(x) \leq f(x_0).$$

Remark 10.3 (Continuity)**.** If f is lower and upper semicontinuous at a point x_0, then f is continuous at x_0. In fact:

$$f(x_0) \leq \liminf_{x \to x_0} f(x) \leq \limsup_{x \to x_0} f(x) \leq f(x_0).$$

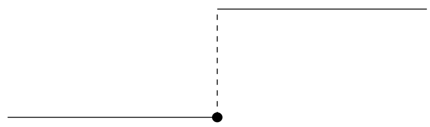


Figure 11: `Example of a lower semicontinuous function.`

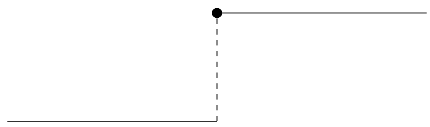


Figure 12: `Example of an upper semicontinuous function.`

Lemma 10.1 (Infimum and supremum of semicontinuous functions)**.** The supremum of continuous functions is lower semicontinuous while the infimum of semicontinuous functions is upper semicontinuous.

Example 10.1. Consider $f_n(x) = x^n$ defined on $[0,1]$ and take the infimum among $n \in \mathbb{N}$. The infimum takes the value 0 in $[0,1)$ but it is 1 at 1, then it is upper semicontinuous.

Proof. Let f_ν a family of continuous functions for any $\nu \in \mathcal{V}$ and let $f(x) := \inf_{\nu\in\mathcal{V}} f_\nu(x)$. Note that $f(x) \leq f_\nu(x)$ for any $\nu \in \mathcal{V}$, hence

$$\limsup_{x\to x_0} f(x) = \limsup_{x\to x_0} \inf_{\nu\in\mathcal{V}} f_\nu(x) \leq \inf_{\nu\in\mathcal{V}} \limsup_{x\to x_0} f_\nu(x) \leq \inf_{\nu\in\mathcal{V}} f_\nu(x_0) = f(x_0).$$

In the same way, one could prove the property for the supremum. □

We conclude recalling the following notions, very useful in viscosity theory.

Definition 10.2 (Lower and upper semicontinuous envelopes)**.** Let $f : \Omega \to \mathbb{R}^n$ (discontinuous) function, then:

1. The *lower semicontinuous envelope* of f is defined as

$$\begin{aligned} f_*(x) &:= \inf\{g(x) \,|\, g \geq f \text{ and } g \text{ lower semicontinuous}\} \\ &= \liminf_{\varepsilon\to 0^+}\{f(y) \,|\, |x-y| \leq \varepsilon\}; \end{aligned}$$

2. The *upper semicontinuous envelope* of f is defined as

$$\begin{aligned} f^*(x) &:= \sup\{g(x) \,|\, g \leq f \text{ and } g \text{ upper semicontinuous}\} \\ &= \limsup_{\varepsilon\to 0^+}\{f(y) \,|\, |x-y| \leq \varepsilon\}. \end{aligned}$$

Exercise 10.1. *Given* $f(x) = \sin\frac{1}{x}$ *on* $[0,+\infty)$*, find* $f^*(x)$ *and* $f_*(x)$*.*

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