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Exercise 3 for the lecture

Numerics IV

SoSe 2012

Due: till Wednesday, November 14, 2012, 14 o'clock

Problem 1 (2 TP)
Suppose $\Gamma \subset \mathrm{R}^{n+1}$ is a C^2-hypersurface, given as the graph of a function $v \in C^2(\Omega)$, i.e. $\Gamma = \{(x, v(x)) \mid x \in \Omega\}$. Let $\nu = \frac{1}{\sqrt{1+|\nabla v|^2}} \begin{pmatrix} \nabla v \\ -1 \end{pmatrix}$. Show that then

$$H(x, v(x)) = \nabla \cdot \left(\frac{\nabla v(x)}{\sqrt{1 + |\nabla v|^2}} \right).$$

Problem 2 (4 + 2 + 4 extra TP)
Suppose $\Gamma \subset \mathrm{R}^{n+1}$ is a C^2-hypersurface, parametrized by a mapping $X \in C^2(V, \mathrm{R}^{n+1})$, $V \subset \mathrm{R}^n$ open, i.e. $\Gamma = X(V)$ and $\mathrm{rank} DX = n$ for all $\theta \in V$. Then we have the following formulae for the tangential gradient of a function f and the mean curvature vector $H\nu$:

$$\nabla_\Gamma f = \sum_{i,j=1}^{n} g^{ij} \frac{\partial (f \circ X)}{\partial \theta_j} \frac{\partial X}{\partial \theta_i}, \tag{1}$$

$$H\nu = -\frac{1}{\sqrt{g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial \theta_i} \left(g^{ij} \sqrt{g} \frac{\partial X}{\partial \theta_j} \right), \tag{2}$$

where $(g_{ij}) = \left(\frac{\partial X}{\partial \theta_i} \cdot \frac{\partial X}{\partial \theta_j} \right)$, $i, j = 1, \dots, n$, $(g^{ij}) = (g_{ij})^{-1}$ and $g = \det(g_{ij})$.

a) Prove (1) for $n \in \mathrm{N}$.

b) Prove (2) for $n = 1$.

c) Prove (2) for $n \in \mathrm{N}$ (Zusatzaufgabe).

Problem 3 (3 TP)
Assume $\Gamma \subset \mathrm{R}^{n+1}$ is a C^2-hypersurface and additionally compact and ϕ a function, that is continously differentiable in a neighbourhood of Γ with values in R^{n+1}. Prove that

$$\int_\Gamma H\nu \cdot \phi \,\mathrm{d}A = \int_\Gamma \nabla_\Gamma x \cdot \nabla_\Gamma \phi \,\mathrm{d}A.$$