Department of Mathematics & Computer Science
Freie Universität Berlin
Prof. Dr. Ralf Kornhuber, Maren-Wanda Wolf

Exercise 5 for the lecture

Numerics IV

SoSe 2012

Due: till Wednesday, December 5, 2012, 14 o'clock

Problem 1 (3 TP + 1 TP)
Consider the parametric curve shortening flow equation

$$X_t - \frac{1}{|X_\theta|}\left(\frac{X_\theta}{|X_\theta|}\right)_\theta = 0 \qquad (\theta,t) \in I \times (0,T) \tag{1}$$
$$X(\cdot,0) = X_0 \qquad \theta \in I$$
$$X(\theta,t) = X(\theta+2\pi,t) \qquad (\theta,t) \in I \times (0,T)$$

and the spatial discretisation

$$\int_I |X_{h,\theta}| X_{h,t} \cdot v + \int_I |X_{h,\theta}|^{-1} X_{h,t} \cdot v = 0 \quad \forall v \in S_h, \tag{2}$$

with $S_h = \{v \in C(I,\mathrm{R}^2) \mid v_{|I_j} \text{ linear}, \ v(0) = v(2\pi)\}$, $I = \bigcup_{j=1}^n I_j$, $I_j = [\theta_{j-1},\theta_j]$, $\theta_j = jh$, $j = 0,\dots,N$ and $h = 2\pi/N$, which is based on the weak formulation of (1).

a) Show that, inserting the basis representation $X_h(\theta,t) = \sum_{j=1}^N X_j(t)\lambda_j(\theta)$, (2) can be rewritten as the following system of odes

$$\frac{1}{6}q_j\dot{X}_{j-1} + \frac{1}{3}(q_j+q_{j+1})\dot{X}_j + \frac{1}{6}q_{j+1}\dot{X}_{j+1} = \frac{X_{j+1}-X_j}{q_{j+1}} - \frac{X_j - X_{j-1}}{q_j}, \quad j = 1,\dots,N, \tag{3}$$

where $q_j = |X_j - X_{j-1}|$.

b) Mass lumping (inexact integration) leads to the simplified system

$$\frac{1}{2}(q_j+q_{j+1})\dot{X}_j = \frac{X_{j+1}-X_j}{q_{j+1}} - \frac{X_j - X_{j-1}}{q_j}, \quad j = 1,\dots,N. \tag{4}$$

Which quadrature rule was used here?

Problem 2 (2 TP + 5 PP + 3 PP + 3 PP)
Consider the parametric curve shortening flow given by (1) and the spatial discretisation given by (4). For time discretisation either apply an implicit or an explicit euler scheme with time step $\tau < 0$.

a) Let $t_m = m\tau$, $m = 0, \ldots, M$, $\tau = T/M$. Derive the algebraic systems to be solved in each time step.

b) Implement the explicit and implicit discrete scheme in MATLAB as `function [X, t] = CurveShorteningImplicit (N, tau, T, X0)` and `function [X, t] = CurveShorteningExplicit (N, tau, T, X0)`, where `N`, `tau`, `T`, and `X0` denote the number of nodes in the space grid, the time step size, the final time $T = 0.5$ and the initial value, the unit circle given as function from I to R^2, respectively.

c) Find out numerically, whether both the explicit and the implicit scheme preserves decreasing (approximate) length with or without conditions on τ.
Illustrate your findings by plotting the approximate length over time for suitable different time steps τ.

d) Plot the discretisation error

$$\max_{[0,T]} \|X - X_h\|_{L^2(I)} + \left(\int_0^T \|X_\theta - X_{h\theta}\|^2_{L^2(I)}\, dt \right)^{\frac{1}{2}}$$

of the implicit scheme over the grid size h, for $N = 10, \ldots, 50$ and $M = 50$.

Problem 3 (4 TP)
Let $X_h(\theta) = \sum_{j=1}^{N} X_j \lambda_j(\theta)$ with vectors $X_j \in \mathrm{R}^2$ be the basis representation of the linear finite element approximation of a curve parametrization $X : I \to \mathrm{R}^2$. Derive a discretisation of curve shortening flow as the gradient flow of the discrete curve length, i.e. differentiate

$$\int_I \|X_{h,\theta}\|$$

with respect to the vectors X_j and derive a system $X_{h,t} = -DX_h$ from this. Compare the resulting discretisation with (3) and (4) from problem 1.