Fachbereich Mathematik & Informatik
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Example Sheet 1

Numerik IVc - Stochastic Processes

Wintersemester 2012

'The different branches of mathematics: ambition, distraction, uglification, and derision.'
Lewis Carroll

Hand in until: Tuesday, 06.11.12, 12:15pm

Discussion in class: Wednesday, 07.11.12, 10:15pm

Exercise 1. (Gaussian random variables)
Let X_1, X_2 be two Gaussian random variables with joint distribution $N(\mu, \Sigma)$, where $\mu = (\mu_1, \mu_2)^T$ and $\Sigma = \begin{pmatrix} a & b \\ b & c \end{pmatrix} > 0$. The density[1] of the joint distribution with respect to the Lebesgue measure on $\mathbb{R}^2$ is given by

$$\rho(x) = \frac{1}{\sqrt{\det(2\pi\Sigma)}} \exp\left(-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\right), \quad x \in \mathbb{R}^2.$$

Show that the conditional probability $\mathbb{P}(X_1 \in A | X_2 = a)$ has a density $\rho_1(x_1, a)$ with respect to the Lebesgue measure on $\mathbb{R}$, and compute $\rho_1(x_1, a)$.

Exercise 2. (Brownian Motion)
Write a `matlab`-program which generates $M = 1000$ realizations of the stochastic process $X_n^{\Delta t}$ with $X_0^{\Delta t} = 0$ and

$$X_{n+1}^{\Delta t} = X_n^{\Delta t} + \sqrt{\Delta t}\xi_{n+1},$$

where (i) $\xi_k \sim N(0,1)$ are iid gaussian random variables and (ii) ξ_k are iid with $\mathbb{P}(\xi_k = \pm 1) = \frac{1}{2}$.

a) For $\Delta t = 1/N$ with $N \in \{100, 1000, 10000\}$, plot a histogram of the distribution of X_N. What do you see?

b) Estimate $\mathbb{E}(\inf_n\{n\Delta t : X_n^{\Delta t} \geq 0.5\})$, i.e. the average time it takes until $X_n^{\Delta t} \geq 0.5$ for the first time, using your matlab program.

c) Now consider the process

$$X_{n+1}^{\Delta t} = X_n^{\Delta t} + (\Delta t)^\alpha \xi_{n+1}, \qquad X_0^{\Delta t} = 0$$

for $\alpha = 0.4$ and $\alpha = 0.6$, and Δt as in (a). Compare the behaviour of the distribution of $X_N^{\Delta t}$ to the one observed in (a). What do you conclude?

[1] A measure ν is said to have density f with respect to another measure μ on the measure space $(\Omega, \mathcal{F}, \mu)$ if $\nu(A) = \int_A f d\mu \quad \forall A \in \mathcal{F}$.

Exercise[2] **3*.** (Characterization of Markov Processes)
Let $(X_t)_{t \in I}$ be a stochastic process adapted to a filtration $\mathcal{F}_t$. Let $\mathcal{Z}_t = \sigma\{X_s | s \geq t\}$ be the σ-algebra corresponding to the 'future' of (X_t). Show: (X_t) is a Markov Process if and only if for all $t \in I$, all $A \in \mathcal{F}_t$ and all $B \in \mathcal{Z}_t$:

$$\mathbb{P}(A \cap B | X_t) = \mathbb{P}(A | X_t)\mathbb{P}(B | X_t),$$

i.e. if 'past' and 'future' are independent if conditioned on the 'present' (*Hint: If* (X_t) *is a Markov process, then one can show the statement using properties of the conditional expectation and the Markov property. For the other direction, try to show that* $\mathbb{P}(A|\mathcal{F}_t) = \mathbb{P}(A|X_t)$ *for all* $A \in \mathcal{Z}_t$.).

[2] Exercises marked with * are not relevant for the exam.