

Example Sheet 2

Numerik IVc - Stochastic Processes

Wintersemester 2012

*'If people do not believe that mathematics is simple,
it is only because they do not realize how complicated life is.'*

John v. Neumann

Hand in until: Tuesday, 20.11.12, 12:15pm

Discussion in class: Wednesday, 21.11.12, 10:15pm

Exercise 1. (Brownian motion)

Let $(B_t)_{t \geq 0}$ be a 1-dimensional Brownian motion. Show, using the definition given in the lecture, the following properties:

- $\mathbb{E}(B_t) = 0$ and $\mathbb{E}(B_t B_s) = \min(s, t) \forall s, t \in \mathbb{R}$.
- $\mathbb{E}[(B_t - B_s)^2] = \mathbb{E}(B_{t-s}^2) = |t - s| \forall s, t \in \mathbb{R}$
- Show that if B_t is a Brownian motion, then $\alpha^{-1/2} B_{\alpha t}$ is also a Brownian motion for any $\alpha > 0$ (scale invariance).
- Let λ^1 be the Lebesgue-measure on \mathbb{R} and let $K \subset \mathbb{R}$ be such that $\lambda^1(K) = 0$. Show that the total length of time that B_t spends in K is zero.¹

Exercise 2. (Brownian Bridges)

Write a `matlab` script to generate $M = 1000$ samples of a Brownian bridge $BB_t = W_t - tW_1$, where W_t is Brownian motion.

- Use the forward integration method discussed in the lecture, i.e.

$$Y_{n+1}^{\Delta t} = Y_n^{\Delta t} \left(1 - \frac{\Delta t}{1 - n\Delta t} \right) + \sqrt{\Delta t} \xi_{n+1}, \quad Y_0^{\Delta t} = 0$$

with $\xi_n \sim N(0, 1)$ iid. Use $\Delta t = 0.001$. Plot a histogram of $Y_N^{\Delta t}$ ($N = 1000$) and estimate $\mathbb{P}(Y_N^{\Delta t} \in [-\epsilon, \epsilon])$ for $\epsilon = 0.01$ and $\epsilon = 0.001$, using your `matlab` program.

- Now create $M = 1000$ samples of the KL-expansion of Brownian motion:

$$BB_t = \sqrt{2} \sum_{k=1}^m \eta_k \frac{\sin(k\pi t)}{k\pi}$$

where $\eta_k \sim N(0, 1)$ iid. Use $\Delta t = 0.01$. For the number m of basis functions use $m \in \{1, 2, 10, 50, 100\}$. Why does it not help to choose $m > (\Delta t)^{-1}$? Plot the variance $\mathbb{E}(BB_t^2)$ versus time t for $t \in [0, 1]$ and compare with theoretical expectations. How does $\mathbb{E}(BB_t^2)$ behave if m is changed?

¹This result extends to the n -dimensional case. It demonstrates that the so-called *Green measure* associated with B_t is absolutely continuous with respect to λ^n .

Exercise 3. (Quadratic Variation)

If $X_t(\cdot) : \Omega \rightarrow \mathbb{R}$ is a continuous stochastic process, then for $p > 0$ the p 'th variation process is defined by

$$\langle X, X \rangle_t^{(p)}(w) = \lim_{\Delta t_k \rightarrow 0} \sum_{t_k \leq t} |X_{t_{k+1}}(w) - X_{t_k}(w)|^p \quad (\text{limit in probability})$$

where $0 = t_1 < t_2 < \dots < t_n = t$ and $\Delta t_k = t_{k+1} - t_k$. In particular, if $p = 1$ this process is called the *total variation process* and if $p = 2$ it is called the *quadratic variation process*.

a) Let B_t be Brownian motion. Show that the quadratic variation is simply

$$\langle B, B \rangle_t^{(2)}(w) = t \quad \text{a.s.}$$

Proceed as follows: Define $\Delta B_k = B_{t_{k+1}} - B_{t_k}$ and put $Y(t, w) = \sum_{t_k \leq t} (\Delta B_k(w))^2$. Show that

$$\mathbb{E} \left[\left(\sum_{t_k \leq t} (\Delta B_k)^2 - t \right)^2 \right] = 2 \sum_{t_k \leq t} (\Delta t_k)^2$$

and deduce that $Y(t, \cdot) \rightarrow t$ in $L^2(P)$ as $\Delta t_k \rightarrow \infty$ (P is the probability measure of B_t).

b) Use a) to prove that a.a. paths of Brownian motion do not have a bounded total variation on $[0, t]$ (*Hint: Try to show that a continuous real function with bounded total variation has zero quadratic variation*).