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Example Sheet 2

# Numerik IVc - Stochastic Processes

Wintersemester 2012

*'If people do not believe that mathematics is simple,*
*it is only because they do not realize how complicated life is.'*
John v. Neumann

**Hand in until:** Tuesday, 20.11.12, 12:15pm

**Discussion in class:** Wednesday, 21.11.12, 10:15pm

**Exercise 1.** (Brownian motion)
Let $(B_t)_{t\geq 0}$ be a 1-dimensional Brownian motion. Show, using the definition given in the lecture, the following properties:

a) $\mathbb{E}(B_t) = 0$ and $\mathbb{E}(B_t B_s) = \min(s,t)\ \forall s,t \in \mathbb{R}$.

b) $\mathbb{E}[(B_t - B_s)^2] = \mathbb{E}(B^2_{t-s}) = |t-s|\ \forall s,t \in \mathbb{R}$

c) Show that if $B_t$ is a Brownian motion, then $\alpha^{-1/2}B_{\alpha t}$ is also a Brownian motion for any $\alpha > 0$ (scale invariance).

d) Let $\lambda^1$ be the Lebesgue-measure on $\mathbb{R}$ and let $K \subset \mathbb{R}$ be such that $\lambda^1(K) = 0$. Show that the total length of time that $B_t$ spends in $K$ is zero.[1]

**Exercise 2.** (Brownian Bridges)
Write a `matlab` script to generate $M = 1000$ samples of a Brownian bridge $BB_t = W_t - tW_1$, where $W_t$ is Brownian motion.

a) Use the forward integration method discussed in the lecture, i.e.

$$Y^{\Delta t}_{n+1} = Y^{\Delta t}_n \left(1 - \frac{\Delta t}{1 - n\Delta t}\right) + \sqrt{\Delta t}\xi_{n+1}, \quad Y^{\Delta t}_0 = 0$$

with $\xi_n \sim N(0,1)$ iid. Use $\Delta t = 0.001$. Plot a histogram of $Y^{\Delta t}_1$ and estimate $\mathbb{P}(Y^{\Delta t}_1 \in [-\epsilon, \epsilon])$ for $\epsilon = 0.01$ and $\epsilon = 0.001$, using your `matlab` program.

b) Now create $M = 1000$ samples of the KL-expansion of Brownian motion:

$$BB_t = \sqrt{2}\sum_{k=1}^{m} \eta_k \frac{\sin(k\pi t)}{k\pi}$$

where $\eta_k \sim N(0,1)$ iid. For the number $m$ of basis functions use $m \in \{1, 2, 10, 50, 100\}$. Plot the variance $\mathbb{E}(BB^2_t)$ versus time $t$ for $t \in [0,1]$ and compare with theoretical expectations. How does $\mathbb{E}(BB^2_t)$ behave if $m$ is changed?

[1] This result extends to the $n$-dimensional case. It demonstrates that the so-called *Green measure* associated with $B_t$ is absolutely continuous with respect to $\lambda^n$.

**Exercise 3.** (Quadratic Variation)
If $X_t(\cdot) : \Omega \to \mathbb{R}$ is a continuous stochastic process, then for $p > 0$ the *p'th variation process* is defined by

$$\langle X, X\rangle_t^{(p)}(w) = \lim_{\Delta t_k \to 0} \sum_{t_k \leq t} |X_{t_{k+1}}(w) - X_{t_k}(w)|^p \quad \text{(limit in probability)}$$

where $0 = t_1 < t_2 < \ldots < t_n = t$ and $\Delta t_k = t_{k+1} - t_k$. In particular, if $p = 1$ this process is called the *total variation process* and if $p = 2$ it is called the *quadratic variation process.*

a) Let $B_t$ be Brownian motion. Show that the quadratic variation is simply

$$\langle B, B\rangle_t^{(2)}(w) = t \quad \text{a.s.}$$

Proceed as follows: Define $\Delta B_k = B_{t_{k+1}} - B_{t_k}$ and put $Y(t, w) = \sum_{t_k \leq t} (\Delta B_k(w))^2$. Show that

$$\mathbb{E}\Big[\Big(\sum_{t_k \leq t} (\Delta B_k)^2 - t\Big)^2\Big] = 2 \sum_{t_k \leq t} (\Delta t_k)^2$$

and deduce that $Y(t, \cdot) \to t$ in $L^2(P)$ as $\Delta t_k \to \infty$ ($P$ is the probability measure of $B_t$).

b) Use a) to prove that a.a. paths of Brownian motion do not have a bounded total variation on $[0, t]$ *(Hint: Try to show that a continuous real function with bounded total variation has zero quadratic variation).*