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Example Sheet 3

Numerik IVc - Stochastic Processes

Wintersemester 2012

'A mathematician is a device for turning coffee into theorems.'
Paul Erdos

Hand in until: Tuesday, 04.12.12, 12:15pm

Discussion in class: Wednesday, 05.12.12, 10:15pm

Exercise 1. (Itô Integrals)
In the following B_t always denotes Brownian motion in $\mathbb{R}$, with $B_0 = 0$ a.s., and $f, g \in \mathcal{V}(0,t)$ are Itô-integrable processes.

a) Prove directly from the definition of the Itô integral that $\int_0^t s dB_s = tB_t - \int_0^t B_s ds$.

b) Use the Itô formula to calculate $I_t = \int_0^t B_s dB_s$.

c) Show that the Itô integral is a martingale, i.e. $\mathbb{E}\left[\int_0^t f(s,\omega)dB_s(\omega)\right] = 0$.

d) Show that $\int_s^t f(u,\omega)dB_u(\omega)$ is $\mathcal{F}_t$-measurable for all $s \leq t$.

Exercise 2. (Ornstein-Uhlenbeck-process)
Let $X_t : \Omega \to \mathbb{R}^n$ be a stochastic process in $\mathbb{R}^n$, and let W_t be Brownian motion in $\mathbb{R}^n$. Consider the SDE

$$dX_t = AX_t dt + BdW_t, \quad X_0 = x \tag{1}$$

where A and B are $(n \times n)$-matrices and $x \in \mathbb{R}^n$.

a) Show that the solution of (1) is given by $X_t = e^{At}x + \int_0^t e^{A(t-s)}dW_s$. *(Hint: Consider $f(x,t) = e^{-At}x$ and use the n-dimensional Itô-formula to compute $df(X_t,t)$.)*

b) Convince yourself that if W_t is interpreted as a column vector, one has

$$\mathbb{E}(W_t W_t^T) = \mathbb{E}\left[\left(\int_0^t dW_s\right)\left(\int_0^t dW_{s'}\right)^T\right] = I$$

where I is the identity matrix.

c) Use (a) and (b) to compute $\mu_t = \mathbb{E}(X_t)$ and the covariance matrix Σ with components

$$\Sigma_t^{ij} = \mathbb{E}[(X_t^i - \mu_t^i)(X_t^j - \mu_t^j)]$$

where X_t^i and μ_t^i are the ith components of X_t and μ_t respectively.

*d)** Show that if all eigenvalues of A have a strictly negative real part, one has the asymptotic representation $A\bar{\Sigma} + A^T\bar{\Sigma} = -BB^T$ where $\bar{\Sigma} = \lim_{t\to\infty} \Sigma_t$ *(Hint: Use partial integration).*

Exercise 3. (Ornstein-Uhlenbeck-process continued)
Write a `matlab` script to numerically integrate the Ornstein-Uhlenbeck-process from (2) in $\mathbb{R}^2$:

$$dX_t = AX_t dt + BdW_t, \quad X_0 = x \in \mathbb{R}^2$$

Choose a timestep of $\Delta t = 0.01$, initial conditions $x = (0,0)$, $B = I$ the identity matrix and A as (i) $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, (ii) $A_2 = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$, (iii) $A_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

a) Plot a realization of the numerically integrated process X_t and compare it with a trajectory of the 'deterministic' process ($B = 0$).

b) Explain the qualitative behaviour you observe in (i), (ii) and (iii) with properties of A *(Hint: eigenvalues...).*