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Example Sheet 5

Numerik IVc - Stochastic Processes

Wintersemester 2012

'Obvious is the most dangerous word in mathematics.'
Eric Temple Bell

Hand in until: Tuesday, 15.01.13, 12:15

Discussion in class: Wednesday, 16.01.13, 10:15

Exercise 1. (Heun's method)
Given the SDE

$$dX_t = b(X_t)\,dt + \sigma(X_t)\,dB_t,$$

the Heun method has the form

$$\tilde{X}_{n+1} = X_n + b(X_n)\Delta t + \sigma(X_n)\Delta B_n, \tag{1}$$

$$X_{n+1} = X_n + \frac{1}{2}\left[b(X_n) + b(\tilde{X}_{n+1})\right]\Delta t + \frac{1}{2}\left[\sigma(X_n) + \sigma(\tilde{X}_{n+1})\right]\Delta B_n. \tag{2}$$

a) Use the example

$$dX_t = 2X_t\,dB_t, \quad X_0 = 1,$$

to show that Heun's method is not strongly nor weakly convergent.

b) How do you explain the result of a)?

Exercise 2. (Infinitesimal generator)
Given an Itô diffusion $(X_t)_{t\geq 0}$ on R^n, the infinitesimal generator L is defined by

$$Lf(x) = \lim_{t\searrow 0} \frac{\mathbb{E}_x\left[f(X_t)\right] - f(x)}{t} \tag{3}$$

for functions $f \in C_0^2(\mathbb{R}^n)$. For a process $(X_t)_{t\geq 0}$ with

$$dX_t = b(X_t)\,dt + \sigma(X_t)\,dB_t$$

this generator has the form

$$Lf(x) = \sum_{i=1}^{n} b_i(x)\frac{\partial f}{\partial x_i}(x) + \frac{1}{2}\sum_{i,j=1}^{n} (\sigma\sigma^T)_{i,j}(x)\frac{\partial^2 f}{\partial x_i \partial x_j}(x). \tag{4}$$

Assume that $b_i \in C^1(\mathbb{R}^n)$ and $(\sigma\sigma^T)_{i,j} \in C^2(\mathbb{R}^n)$ for all $i,j = 1,...,n$. Determine the L^2-adjoint operator L^* of L, i.e. find L^* such that

$$\int_{\mathbb{R}^n} (Lf)(x)g(x)\,dx = \int_{\mathbb{R}^n} f(x)(L^*g)(x)\,dx$$

for all $f, g \in C_0^2(\mathbb{R}^n)$ with compact support[1].

[1]The support of a function f on $\mathbb{R}^n$ is defined as $\mathrm{supp} f := \overline{\{x \in \mathbb{R}^n : f(x) \neq 0\}}$.

Exercise 3. (Kolmogorov forward equation)
Given the infinitesimal generator L of an Itô diffusion, the Kolmogorov forward (or Fokker-Planck) equation

$$\frac{\partial}{\partial t}\rho_t = L^* \rho_t \tag{5}$$

describes the propagation of the related probability density function $\rho_t = \rho_t(x)$ in time.
Consider the process

$$X_t = X_0 + B_t, \quad X_0 \sim \mu_0 \tag{6}$$

with B_t denoting Brownian motion in $\mathbb{R}$. Let μ_0 be the uniform distribution on $[-1, 1]$. We are interested in the probability $\mathbb{P}_{\mu_0}(X_T \in A)$ of the process being in a set $A \subset \mathbb{R}$ at time $T > 0$ given the initial distribution μ_0.

a) Use a Monte Carlo scheme to estimate $\mathbb{P}_{\mu_0}(X_T \in A)$ for $T = 1$ and $A = [2, 3]$.

b) Calculate $\mathbb{P}_{\mu_0}(X_T \in A)$ for $T = 1$ and $A = [2, 3]$ by means of the Kolmogorov forward equation. *Hint: For the numerical implementation you need to find an adequate discretization of the state space and the generator. E.g., choose an equidistant grid on* $[-5, 5]$ *with suitable boundary conditions.*