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Example Sheet 6

Numerik IVc - Stochastic Processes

Wintersemester 2012

'If I were to awaken after having slept for a thousand years, my first question would be: Has the Riemann hypothesis been proven?'
David Hilbert

Hand in until: Tuesday, 29.01.13, 12:15pm

Discussion in class: Wednesday, 30.01.13, 10:15pm

Exercise 1. (Continuous Semigroup)
Let $(X_t)_{t\geq 0}$ be a Markov process with semigroup $S_t : K \to K$ generated by $L : K \to K$. Show that $LS_t = S_t L$.

Exercise 2. (Commitor equation)
Let $(X_t)_{t\geq 0}$ be an Itô process on $\mathbb{R}$ with generator L. Let $A = [a_1, a_2] \subset \mathbb{R}$ and $B = [b_1, b_2] \subset R$ be compact with $b_1 > a_2$, and let $C = [a_2, b_1]$. Let τ_A and τ_B be the first hitting times of A and B, and define for any $x \in C$ the commitor function

$$q(x) = \mathbb{P}(\tau_A > \tau_B | X_0 = x).$$

a) Show that the commitor function solves the commitor equation

$$Lq = 0 \ \ \forall x \in C, \quad q|_A = 1, \quad q|_B = 0.$$

(Hint: Rewrite $q(x)$ as a conditional expectation, and use an appropriate Feynman-Kac formula or Dynkin's formula.)

b) Consider now the simple random walk on the lattice with spacing Δx, this is a Markov chain with transition probabilities $\mathbb{P}(X_{n+1} = x \pm \Delta x | X_n = x) = \frac{1}{2}$. Use the Markov property to show that $q(x)$ defined as above satisfies the boundary value problem

$$\frac{1}{2}\Delta_d q = 0 \ \ \forall x \in C, \quad q|_A = 1, \quad q|_B = 0$$

where Δ_d is the finite-difference discretization of the Laplacian on the lattice with spacing Δx. Compare with (a). *(Note that $L_d = P - I = \frac{1}{2}\Delta_d$ is the generator of $(X_n)_{n\in\mathbb{N}}$.)*

Exercise 3. (Metropolis Hastings Algorithm)
Let $(X_t)_{t\geq 0}$ be an Itô process given by the SDE

$$dX_t = -\nabla V(X_t)dt + \sqrt{2\epsilon}dB_t \tag{1}$$

with $\epsilon > 0$ a real parameter, and the potential function $V(x) = (x^2 - 1)^2$.

a) Compute the invariant density ρ_∞ up to the normalization constant.

b) One way to sample ρ_∞ numerically is by using a long trajectory. The hope is that the system converges to ρ_∞ fast enough such that one eventually ends up sampling ρ_∞. We want to try this out. Numerically integrate (1) with $X_0 = -1$ using forward Euler for $\epsilon = 0.15$ with $\Delta t = 0.05$. Use $N = 10^4, 10^5$ and 10^6 steps. Create a histogram of all the positions visited and compare with (a)[1]. What are the problems you observe? What happens if you change Δt?

c) The Metropolis-Hastings algorithm can be used to sample distributions which are known up to their normalization constant. The algorithm creates a Markov chain $(X_n)_n$ which has the chosen distribution (here ρ_∞) as invariant distribution. It consists of two steps:

   (i) Proposal step: Starting from X_n, create a new state $\tilde{X}_{n+1}$ drawn from some proposition density $T(X_n, \tilde{X}_{n+1})$.

   (ii) Acceptance step: Compute the Metropolis-Hastings ratio

$$r(X_n, \tilde{X}_{n+1}) = \frac{T(\tilde{X}_{n+1}, X_n)\rho_\infty(\tilde{X}_{n+1})}{T(X_n, \tilde{X}_{n+1})\rho_\infty(X_n)}$$

   and accept the proposition with probability $\min\Big(1, r(X_n, \tilde{X}_{n+1})\Big)$. In this case set $X_{n+1} = \tilde{X}_{n+1}$, otherwise $X_{n+1} = X_n$.

   Implement the Metropolis-Hastings algorithm using the Euler discretization of (1) as the proposal step. Convince yourself that

$$\tilde{X}_{n+1} \sim \mathcal{N}\left(X_n - \Delta t \nabla V(X_n), 2\epsilon \Delta t\right)$$

   follows and calculate $T(X_n, \tilde{X}_{n+1})$. Use the result of (a) for the computation of $r(X_n, \tilde{X}_{n+1})$. Use the same parameters as in (b) for the implementation. Create a histogram of the positions X_n and compare your results with (a) and (b). What do you observe? Use your algorithm to try out different values for Δt and try to find the optimal choice. What happens if Δt becomes either much larger or much smaller?

d) Repeat (c), but this time use $\tilde{X}_{n+1} \sim \mathcal{N}(0, 0.5^2)$ for the proposal step.

[1] The comparison of two probability distributions μ_1 and μ_2 with densities ρ_1 and ρ_2 is best done by inspecting $d(\mu_1, \mu_2)$ where $d(\cdot,\cdot)$ is an appropriate distance function, two viable options are $d(\mu_1, \mu_2) = \|\mu_1 - \mu_2\|_{L_1}$ or $d(\mu_1, \mu_2) = D_{KL}(\mu_1 || \mu_2) = \int \log\left(\frac{\rho_1(x)}{\rho_2(x)}\right)\rho_1(x)dx$.