Fachbereich Mathematik und Informatik
Freie Universität Berlin
Prof. Dr. Carsten Gräser

Exercise 2 for the lecture

Fast solvers for nonsmooth PDEs

Winter term 2014

http://numerik.mi.fu-berlin.de/wiki/WS_2014/FastSolvers.php

Due: Thu, 2014-11-13, 10:15

Problem 1 (8 theory points)
Let H be a Hilbert space and $f : H \to \mathbb{R}$ Gaṫeaux-differentiable. f is called strongly monotone with modulus $\mu > 0$ if

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) - \lambda(1-\lambda)\frac{\mu}{2}\|x-y\|^2 \qquad \forall x, y \in H, \lambda \in [0,1].$$

a) Show that f is convex, iff

$$f(x) - f(y) \geq \langle Df(y), x-y \rangle \qquad \forall x, y \in H. \tag{1}$$

b) Show that f is convex, iff $Df : H \to H'$ is monotone, i.e.,

$$\langle Df(x) - Df(y), x-y \rangle \geq 0 \qquad \forall x, y \in H. \tag{2}$$

c) Show that f is strongly convex with modulus $\mu > 0$ with , iff

$$f(x) - f(y) \geq \langle Df(y), x-y \rangle + \frac{\mu}{2}\|x-y\|^2 \qquad \forall x, y \in H. \tag{3}$$

d) Show that f is strongly convex with modulus $\mu > 0$, iff $Df : H \to H'$ is strongly monotone, i.e.,

$$\langle Df(x) - Df(y), x-y \rangle \geq \mu\|x-y\|^2 \qquad \forall x, y \in H. \tag{4}$$

Problem 2 (8 theory points)
Let V a reflexive Banach-space, $J_0 : V \to \mathbb{R}$ convex and Gâteaux-differentiable, $\varphi : V \to \mathbb{R}$ convex and $K \subset V$ convex. Show that the minimization problem

$$u \in K : \qquad J(u) \leq J(v) \qquad \forall v \in K$$

for $J = J_0 + \varphi$ is equivalent to the variational inequality

$$u \in K : \qquad \langle J_0'(u), v-u \rangle + \varphi(v) - \varphi(u) \geq 0 \qquad \forall v \in K.$$

What happens if $J = \frac{1}{2}a(\cdot,\cdot) - l(\cdot)$ is a quadratic functional? Hints: Consider $\tilde{v} = (1-\lambda)u + \lambda v$. Show for convex $f \in C^1(\mathbb{R})$

$$f(y) - f(x) \geq f'(x)(y-x) \qquad \forall x, y \in \mathbb{R}.$$