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Exercise 3 for the lecture

Fast solvers for nonsmooth PDEs

Winter term 2014

http://numerik.mi.fu-berlin.de/wiki/WS_2014/FastSolvers.php

Due: Thu, 2014-11-20, 10:15

Problem 1 (4 theory points)
Let $\underline{\psi}, \overline{\psi} \in H^1(\Omega)$ with $\underline{\psi} \leq \overline{\psi}$. Show that the set

$$K = \{v \in H^1(\Omega) \,|\, \underline{\psi} \leq v \leq \overline{\psi} \text{ a.e.}\}$$

is convex and closed in $H^1(\Omega)$ and that its characteristic functional χ_K is convex, proper, and l.s.c.. Furthermore show that for $\underline{\psi} \neq \overline{\psi}$ the set K is unbounded and χ_K is not coercive.

Problem 2 (4 theory points)
Let H be a Hilbert space, $a(\cdot,\cdot) : H \times H \to \mathbb{R}$ a symmetric, elliptic bilinear form, $l \in H'$, and $K \subset H$ closed, convex, and nonempty. Show that the minimization problem

$$u \in K : \qquad J(u) \leq J(v) \qquad v \in K$$

for $J(v) = \frac{1}{2}a(v,v) - l(v)$ has a unique solution.

Problem 3 (4 theory points)
Let Ω be a bounded polyhedral domain, $\mathcal{T}$ a conforming triangulation of Ω by simplices, $\mathcal{S}_\mathcal{T}$ the space of piecewise linear finite element functions on $\mathcal{T}$ and $I_\mathcal{T} : C(\Omega) \to \mathcal{S}_\mathcal{T}$ the nodal interpolation operator. For a convex, proper, l.s.c. functional $\Phi : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ define

$$\varphi : L^1(\Omega) \to \mathbb{R} \cup \{\infty\}, \qquad \varphi(v) = \int_\Omega \Phi(v(x))\,dx.$$

a) Show that φ is proper and convex.

b) Show that the lumped approximation

$$\varphi_\mathcal{T} : \mathcal{S}_\mathcal{T} \to \mathbb{R} \cup \{\infty\}, \qquad \varphi_\mathcal{T}(v) = \int_\Omega I_\mathcal{T}(\Phi \circ v)(x)\,dx$$

is proper and convex and satisfied $\varphi(v) \leq \varphi_\mathcal{T}(v)$ for all $v \in \mathcal{S}_\mathcal{T}$.