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Exercise 4 for the lecture

Fast solvers for nonsmooth PDEs

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`http://numerik.mi.fu-berlin.de/wiki/WS_2014/FastSolvers.php`

Due: Thu, 2014-11-27, 10:15

Problem 1 (8 theory points)
Let $A \in \mathbb{R}^{n\times n}$ be s.p.d. and $b, \psi \in \mathbb{R}^n$. For the minimization problem

$$u \in K: \qquad J(u) \leq J(v) \qquad \forall v \in K$$

with $J(v) = \frac{1}{2}\langle Av, v\rangle - \langle b, v\rangle$ and $K = \{v \in \mathbb{R}^n \mid \psi \leq v\}$ and the lower/diagonal/upper splitting $A = L + D + R$ we consider the projected Gauß–Seidel method given by

$$u^{\nu+1} = (D + L + \partial\chi_K)^{-1}(b - Ru^\nu) = u^\nu + F(u^\nu).$$

a) Show that u solves the minimization problem iff $F(u) = 0$ for F as defined above.

b) Show that $F : K \to K$ is Lipschitz-continuous.

c) For $v \in \mathbb{R}^n$ we denote by $\mathcal{A}(v) = \{i \in \mid \psi_i = v_i\}$ and $\mathcal{I}(v) = \{1, \dots, n\} \setminus \mathcal{A}(v)$ be the sets of active and inactive indices. Now let $\mathcal{J} \subset \{1, \dots, n\}$ be any index set and $U(\mathcal{J}) = \{v \in \mathbb{R}^n \mid \mathcal{I}(v + F(v)) = \mathcal{J}\}$, i.e., for all v in $U(\mathcal{J})$ the active set is the same after application of one Gauß–Seidel step. Show that F is affine linear on $U(\mathcal{J})$.

Problem 2 (4 theory points)
For $M \in \mathbb{R}^{m\times n}$ the Moore–Penrose pseudoinverse $M^+ \in \mathbb{R}^{n\times m}$ is the uniquely determined a matrix with

$$MM^+M = M, \qquad M^+MM^+ = M^+, \qquad (MM^+)^T = MM^+, \qquad (M^+M)^T = M^+M.$$

For $M \in \mathbb{R}^{n\times n}$ and an index set $\mathcal{I} \subset \{1, \dots, n\}$ we denote by $M_\mathcal{I} \in \mathbb{R}^{n\times n}$ the matrix with $(M_\mathcal{I})_{i,j} = M_{i,j}$ for $i, j \in \mathcal{I}$ and $(M_\mathcal{I})_{i,j} = 0$ else. Now let $A \in \mathbb{R}^{n\times n}$ and $\mathcal{I} \subset \{1, \dots, n\}$.

a) Show that $A_\mathcal{I} = I_\mathcal{I} A I_\mathcal{I}$

b) Assume that the $|\mathcal{I}| \times |\mathcal{I}|$ submatrix of A indexed by $\mathcal{I}$ is regular. Show that $(A_\mathcal{I})^+ = (A_\mathcal{I} + I - I_\mathcal{I})^{-1} - I + I_\mathcal{I}$.