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Exercise 5 for the lecture

Fast solvers for nonsmooth PDEs

Winter term 2014

http://numerik.mi.fu-berlin.de/wiki/WS_2014/FastSolvers.php

Due: Thu, 2014-12-04, 12:15

Problem 1 (12 programming + 4 theory points)
For $A \in \mathbb{R}^{n\times n}$ s.p.d., $b, \underline{\psi}, \overline{\psi} \in \mathbb{R}^n$ with $\underline{\psi} \leq \overline{\psi}$, and $K = \{v \in \mathbb{R}^n \,|\, \underline{\psi} \leq \overline{\psi}\}$ we consider the minimization problem

$$u = \arg\min_{v\in K} \frac{1}{2}\langle Av, v\rangle - \langle b, v\rangle.$$

a) Implement a projected Gauß–Seidel increment operator as

```
y = pgs_inc(A,r,lower,upper)
```

computing $y = (D + L + \partial\chi_K)^{-1}r$ for a $D + L + R = A$ splitting of A and `lower` and `upper` representing $\underline{\psi}$ and $\overline{\psi}$, respectively.

b) Implement a projected Gauß–Seidel-step as

```
xnew = pgs(A,b,lower,upper,xold)
```

computing $x_{new} = (D + L + \partial\chi_K)^{-1}(b - Rx_{old})$. Base you implementation on `pgs_inc`.

c) Implement a projected Gauß–Seidel solver as

```
x = pgs_solver(A,b,lower,upper,x0,maxnu)
```

doing `maxnu` Gauß–Seidel iterations returning the last iterate as `x`.

d) Test your method using the test suite provided on the lecture homepage.

e) Illustrate numerically (using a $1d$ example) why the projected Gauß–Seidel method can still be viewed as a smoother for the obstacle problem for the Laplacian.

f) Give a $1d$ example where you can prove that $\|x^\nu - x\|_A \geq 1 - C\frac{\nu}{n}$ where $\|\cdot\|_A$ is the energy norm induced by A.

Take care to have optimal complexity for sparse matrices in all implementations.