

Exercise 9 for the lecture

## FAST SOLVERS FOR NONSMOOTH PDES

Winter term 2014

[http://numerik.mi.fu-berlin.de/wiki/WS\\_2014/FastSolvers.php](http://numerik.mi.fu-berlin.de/wiki/WS_2014/FastSolvers.php)

**Due: Thu, 2015-01-29, 12:15**

**Problem 1** (8 programming + 6 theory points)

For  $A \in \mathbb{R}^{n \times n}$  s.p.d.,  $b, \underline{\psi}, \bar{\psi} \in \mathbb{R}^n$  with  $\underline{\psi} \leq \bar{\psi}$ , and  $K = \{v \in \mathbb{R}^n \mid \underline{\psi} \leq \bar{\psi}\}$  we consider the minimization problem

$$u = \arg \min_{v \in \mathbb{R}^n} J(v), \quad J(v) = J_0(v) + \chi_K(v), \quad J_0(v) = \frac{1}{2} \langle Av, v \rangle - \langle b, v \rangle$$

and a given hierarchy of subspaces  $V_1 \subset \dots \subset V_m = \mathbb{R}^m$ ,  $\dim V_k = n_k$ .

- a) Show that for  $u \in K$  and  $v \in \mathbb{R}^n$  the Euclidean projection  $P_{K-u}$  into  $K - u$  is given by

$$(P_{K-u}(v))_i = \max\{\underline{\psi}_i - u_i, \min\{\bar{\psi}_i - u_i, v_i\}\}.$$

- b) Show that for  $u \in K$  and  $v \in \mathbb{R}^n$  we have

$$\arg \min_{\rho \in \mathbb{R}} J(u + \rho) = \max \left\{ \underline{\delta}, \min \left\{ \bar{\delta}, \frac{\langle b - Au, v \rangle}{\langle Av, v \rangle} \right\} \right\}$$

where

$$\underline{\delta} = \max\{(\underline{\psi} - u - v)_i \mid i = 1, \dots, n\}, \quad \bar{\delta} = \min\{(\bar{\psi} - u - v)_i \mid i = 1, \dots, n\}.$$

- c) Implement a TNNMG step as

$$v = \text{tnnmg\_step}(r, \text{lower}, \text{upper}, AA, PP) .$$

- d) Implement TNNMG solver as

$$x = \text{tnnmg\_solver}(A, b, \text{lower}, \text{upper}, x0, PP, \text{maxnu}).$$

- e) Test your method using the test suite provided on the lecture homepage.  
f) Extend the test suite to allow initial iterates from nested iteration.

- g) Compare iteration history and convergence rate of MMG, TMMG, TNNMG for trivial initial iterates and nested iteration.
- h) Give a  $1d$  example where you can prove that  $\|u^\nu - u\|_A \geq 1 - C \frac{\nu}{n}$  holds for the TNMMG method with bad initial iterates. Illustrate this example numerically and compare with MMG and TMMG and explain the results.

Take care to have optimal complexity for sparse matrices in all implementations.