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Exercise 11 for the lecture

Numerics II

WS 2014/15

Due: till Tuesday, 27. January

Problem 1

Consider the the linear system

$$AU = b \tag{1}$$

with the symmetric positive definite matrix $A \in \mathrm{R}^{n,n}$ and $b \in \mathrm{R}^n$.

a) Compute an upper bound for the convergence rate of the Jacobi method applied to the linear system (1) with the matrix A obtained by a finite difference discretization of the Laplace equation using a uniform grid on $[0,1] \times [0,1]$ given in the lecture.

b) Implement the Jacobi and the Gauß-Seidel methods in `matlab` as

```
function [u, uk] = Jacobi(A, b, u0, tol, uexact)
```

and

```
function [u, uk] = GaussSeidel(A, b, u0, tol, uexact).
```

`u`, `uk`, `A`, `b`, `u0`, `tol`, and `uexact` denote the last iterate, a vector containing all iterates, the system matrix, the right hand side, the initial iterate, the error tolerance, and the exact solution, respectively. The iteration should stop if the energy norm $\|\cdot\|_A = \langle A\cdot, \cdot\rangle^{0.5}$ of the error is smaller than the tolerance. Test your programm with the matrix of part a) and the right hand side $b = AU$ where U is the point wise evaluation of $(x_1 - x_1^2)(x_2 - x_2^2)$ for `u0` $= 0$, `tol` $= 10^{-8}$ and various choices of n. Plot the error over the number of iteration steps and compute the average convergence rate for each choice of n.

Problem 2
The symmetric Gauß-Seidel method for the solution of a linear system with a s.p.d. matrix is obtained by applying one normal Gauß-Seidel step and one Gauß-Seidel step with the components in reversed order alternatingly. Give the iteration matrix and the generated preconditioner of the symmetric Gauß-Seidel method and show that the preconditioner is s.p.d..

Problem 3
Show that

$$\kappa(B^{-1}A) \leq \frac{\mu_1}{\mu_0}$$

holds, if $B \in \mathrm{R}^{n \times n}$ is a prconditioner satisfying

$$\mu_0(Bx, x) \leq (Ax, x) \leq \mu_1(Bx, x) \quad \forall x \in \mathrm{R}^n$$

for some $0 \leq \mu_0, \mu_1 \in \mathrm{R}$.

Problem 4

a) Show that the parallel directional correction method associated with the Euclidean unit vectors e_i is the Jacobi method.

b) Show that the successive directional correction method associated with the Euclidean unit vectors e_i is the Gauß-Seidel method.

Problem 5
Assume that $A \in \mathrm{R}^{n,n}$ is s.p.d.. Prove the convergence of the gradient method

$$x^{k+1} = x^k + \omega^k r^k, \qquad r^k = b - Ax_k, \qquad \omega^k = \frac{\langle r^k, r^k \rangle}{\langle Ar^k, r^k \rangle}.$$