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Exercise 5 for the lecture

Numerics II

WS 2014/15

Due: till Tuesday, 25. November

Problem 1

Compute the Butcher scheme for the collocation method with the supporting points of the Simpson rule.

Problem 2

a) Implement the (possibly implicit) Runge-Kutta method for the linear system

$$x'(t) = Mx(t), \qquad t \in (\mathtt{I(1)}, \mathtt{I(2)}] \qquad x(\mathtt{I(1)}) = \mathtt{x_0}$$

in `matlab` as `function [x, t, k] = RungeKuttaLinear(M, x0, I, tau, b, A)`, where `M`, `x0`, `I`, and `tau` denote the system matrix, the initial value, the time interval and the step size, respectively and the Butcher scheme is given by `b`, `A`. The returned values `x`, `t`, and `k` should contain the solution at each time step, the time steps, and the intermediate vectors k_i for all time steps, respectively.

b) Test your program with the linear initial value problem

$$x'(t) = -\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} x(t), \quad x(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

on the interval $(0, 5]$ with the step sizes $\tau = 10^{-3}, 10^{-2}, 10^{-1}, 1$ for the method of Runge, and the Gauß method of order 4. Plot the discretization error and discuss the numerical results.

c) Evaluate the collocation polynomials u of the Gauß method of order 4 applied to the above problem with step size $\tau = 1$ from the intermediate vectors k_i on a sample grid $\Theta_i = i\frac{\tau}{100}$, $i = 0, \ldots, 100$ and plot the discrete trajectories given by the values of u.

Problem 3
The discrete flux $\Psi^{t+\tau,t}x(t)$ (approximating $x(t+\tau)$) of Runge-Kutta methods for non-autonomous initial value problems

$$x'(t) = f(t, x(t)), \quad t > t_0, \qquad x(0) = x_0$$

with $f : \Omega \to \mathrm{R}^d$ is specified by

$$\Psi^{t+\tau,t}x = x + \tau \sum_{i=1}^{s} b_i k_i, \qquad\qquad k_i = f(t + c_i\tau, x + \tau \sum_{j=1}^{s} a_{ij} k_j).$$

The coefficients are given in the extended Butcher scheme

$$\begin{array}{c|c} c & \mathcal{A} \\ \hline & b^T \end{array}.$$

A Runge-Kutta method is called invariant with respect to autonomization if the discrete flux coincides with the discrete flux $\overline{\Psi}^{\tau}$ of the autonomized system

$$y'(t) = F(y(t)), \quad t > 0, \qquad y(0) = (x_0, t_0)^T$$

with $F : \Omega \times \mathrm{R}^+ \to \mathrm{R}^{d+1}$, $y = (x, t)^T \mapsto F(y) = (f(t, x), 1)^T$, i.e.

$$\overline{\Psi}^{\tau} \begin{pmatrix} x \\ t \end{pmatrix} = \begin{pmatrix} \Psi^{t+\tau,t}x \\ t + \tau \end{pmatrix}.$$

Show that collocation methods are invariant with respect to autonomization.

Problem 4
Prove the following remarks:

a) Let f be dissipativ. Then every fixed point of $x'(t) = f(x)$ is stable.

b) If f satisfies the stronger condition

$$(f(x) - f(y), x - y) \le \mu |x - y|^2 \qquad \mu < 0,$$

then every fixed point is assymptotically stable.