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Exercise 6 for the lecture

Numerics II

WS 2014/15

Due: till Tuesday, 2. December

Problem 1

Consider the initial value problem

$$x'(t) = \frac{\lambda x}{g(x)}, \qquad t > 0, \qquad x(0) = x_0, \tag{1}$$

with $g(0) > 0$ and $\lambda < 0$ and $g \in C^1(\mathrm{R})$.

a) Show that $x^* = 0$ is an asymptotically stable fixed point of (1).

b) Show that the semi-implicit time discretization

$$\frac{x_{k+1} - x_k}{\tau} = \frac{\lambda x_{k+1}}{g(x_k)}$$

is asymptotically stable, in the sense that $\lim_{k\to\infty} x_k = 0$ for all τ and x_0.

Problem 2

Consider the scalar differential equation

$$x' = \lambda(1 - x^2), \qquad \lambda > 0. \tag{2}$$

a) Show that $x_s^* = 1$ is an asymptotically stable and $x_u^* = -1$ an unstable fixed point of (2).

b) Compute the time step restriction for the explicit euler method applied to the linearized problem

$$x' = -2\lambda(x - 1),$$

such that x_s^* is an asymptotically stable fixed point of the resulting discrete linear problem.

c) Is the time step restriction from b) also sufficient to guarantee that x_s^* is an asymptotically stable fixed point of the discrete nonlinear problem, resulting from apllying the explicit euler method to (2)?

Problem 3 (6 PP)
Consider the following nonlinear initial value problem

$$x'(t) = f(x), \qquad t > 0, \qquad x(0) = x_0,$$

with $x(t) \in \mathrm{R}^3$ and

$$f(x) = \begin{pmatrix} -c_1x_1 + c_3x_2x_3 \\ c_1x_1 - c_2x_2^2 - c_3x_2x_3 \\ c_2x_2^2 \end{pmatrix}.$$

a) Implement an implicit Runge-Kutta method for this equation in MATLAB as `function [x, t] = RungeKuttaNewton(f, Df, x0, I, tau, b, A, TOL)`, where `f`, `Df`, `x0`, `I`, and `tau` denote the function implementing the right hand side f, the Jacobian Df, the initial value, the time interval, the step size, and the tolerance for the nonlinear solver respectively and the Butcher scheme is given by `b, A`. The returned values `x`, `t` should contain the solution at each time step and the time steps, respectively. Use the simplified Newton method and the stopping criterion presented in the lecture to solve the nonlinear system

$$F(Z) = Z - \tau \begin{pmatrix} \sum_{j=1}^s a_{1j} f(x + z_j) \\ \vdots \\ \sum_{j=1}^s a_{sj} f(x + z_j) \end{pmatrix} = 0,$$

with $Z = [z_1, \ldots, z_s]^T \in \mathrm{R}^{3s}$.

b) Test your code for the initial value $x_0 = (1, 0, 0)^T$, the time interval $[0, 10]$, and the parameter vector $c = (1, 2, 3)$. Use the implicit Euler method, the implicit midpoint rule and the Gauß-method of order 6 for your tests. Estimate the discretization error by computing a sufficiently good reference solution numerically. Plot the estimated error as a function of the step size for all methods and for an appropiate set of step sizes. What convergence orders do you observe ?

c) Plot the total mass $x_1 + x_2 + x_3$ as function of the time. Show that the continuous flux and every Runge-Kutta scheme preserve the total mass.