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Exercise 8 for the lecture

# Numerics II

WS 2014/15

**Due: till Tuesday, 16. December**

**Problem 1**

Consider the *heat equation*

$$\frac{d}{dt}u(x,t) = \Delta u(x,t) \tag{1}$$

with $u : [a,b] \times \mathrm{R}_0^+ \to \mathrm{R}$, the boundary conditions $u(a,t) = u(b,t) = 0$ and the initial condition $u(x,0) = u_0(x)$. Let there be an equidistant partition $a < x_1 < \ldots < x_n < b$ of the interval $[a,b]$, i.e.,

$$x_i = a + \frac{i(b-a)}{n+1}, \qquad i = 1, \ldots, n.$$

The quantity $h = (b-a)/(n+1)$ is called the *grid size*.

a) Discretize (1) by central difference quotients at the points $x_i$. Write the spatially discrete problem as

$$u_h'(t) = -A_h u_h(t), \qquad u_h(0) = u_{h,0}$$

with $u_h(t) \in \mathrm{R}^n$ and give $u_{h,0}$ and the matrix $A_h$.

b) Show that the explicit Euler scheme is stable for $\tau \leq \frac{1}{2}h^2$. Show that this upper bound is asymptotically sharp for $h \to 0$.

c) Show that there is a functional $E_h : \mathrm{R}^n \to \mathrm{R}$ such that

$$u_h'(t) = -\nabla E_h(u_h(t)).$$

d) Show that $E_h$ is strictly convex.

**Problem 2** (4 PP)
The so called curve shortening flow

$$\begin{aligned} u_t - \frac{1}{|u_x|}\left(\frac{u_x}{|u_x|}\right)_x &= 0 && \text{in } I \times (0,T) \\ u(0,t) &= u(2\pi,t) && \text{in } (0,T) \\ u(\cdot,0) &= u_0, \end{aligned}$$

is obtained as the gradient flow of the functional

$$E(u) = \int_I |u_x| \, dx,$$

describing the length of a closed curve. Here $u : I \times (0,T) \to \mathrm{R}^2$ and $u(\cdot,t)$ describes the position of the curve in $\mathrm{R}^2$ at the time $t$ parametrized over the interval $I = [0, 2\pi]$. To solve this problem numerically we use the space discrete approximation

$$\begin{aligned} U_j' &= \frac{2}{|U_{j+1} - U_j| + |U_j - U_{j-1}|}\left(\frac{U_{j+1} - U_j}{|U_{j+1} - U_j|} - \frac{U_j - U_{j-1}}{|U_j - U_{j-1}|}\right) \\ U_j &= U_{j+N} && \text{for } j = -1, 0, 1 \end{aligned}$$

with $U : (0,T) \to \mathrm{R}^{2\times N}$. Notice that each $U_j$ is a function, mapping $(0,T)$ to $\mathrm{R}^2$ and approximates the value of $u(x_j,t)$ with the equidistant space grid $(x_j)_{j=0,\ldots,N}$.
Following the ideas for problem 1 on exercise 6 a time discretization is given by

$$\frac{1}{\tau}(U_j^{m+1} - U_j^m) = \frac{2}{(|U_{j+1}^m - U_j^m| + |U_j^m - U_{j-1}^m|)}\left(\frac{U_{j+1}^{m+1} - U_j^{m+1}}{|U_{j+1}^m - U_j^m|} - \frac{U_j^{m+1} - U_{j-1}^{m+1}}{|U_j^m - U_{j-1}^m|}\right).$$

Implement the above iteration in MATLAB as `function [u, t] = CurveShortening(N, tau, T, u0)`, where `N`, `tau`, `T`, and `u0` denote the number of nodes in the space grid, the time step size, the final time and the initial value given as function from $I$ to $\mathrm{R}^2$ respectively. Test your program with interesting initial values and appropriate parameters.