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Exercise 9 for the lecture

Numerics II

WS 2014/15

Due: till Tuesday, 13. January

Problem 1

Consider the differential algebraic system

$$\begin{pmatrix} 0 & 0 & 0 \\ c & 0 & -c \\ 0 & 0 & 0 \end{pmatrix} x' = \begin{pmatrix} 1 & 0 & -1 \\ c & -\frac{1}{R} & \frac{1}{R} \\ 0 & 0 & 1 \end{pmatrix} x + \begin{pmatrix} -U \\ 0 \\ 0 \end{pmatrix}, \qquad x(0) = x_0 \tag{1}$$

with $c, R, U > 0$.

a) Rewrite this system in normal form, i.e., as

$$\begin{aligned} y'(t) &= Jy(t) + f, & y(0) &= y_0, \\ Nz'(t) &= z(t) + g, & z(0) &= z_0. \end{aligned}$$

Give f, g, J, N, y_0, z_0, and the transformation $T : x \mapsto (y, z)$.

b) How must x_0 be chosen such that there is a unique solution of (1) ?

Problem 2

Show that the differentiation index of a linear differential algebraic system is invariant under equivalence transformations.

Problem 3

Consider the DAE

$$Nz'(t) = z(t) + f(t) \tag{2}$$

with $f \in C^\infty(\mathrm{R}, \mathrm{R}^d)$. Show that if N is nilpotent of degree ν then ν-fold derivation of (2) leads to an ODE without algebraic constraints. What is the differentiation index of (2) in this case ?

Problem 4
Consider the initial value problem

$$\begin{aligned} y'(t) &= f(y(t), z(t)), & y(0) &= y_0, \\ \epsilon z'(t) &= g(y(t), z(t)), & z(0) &= z_0 \end{aligned}$$

with

$$f(y, z) = yz, \qquad g(y, z) = z - z^3.$$

For which solutions y^*, z^* of $g(y^*, z^*) = 0$ does

$$\max_{\lambda} \mathrm{Re}(\lambda) < 0, \qquad \lambda \in \sigma\left(\frac{dg}{dz}(y^*, z^*)\right)$$

hold? Solve the corresponding DAE system for $\epsilon = 0$ with consistent initial values. Discuss the asymptotic behavior of the singularly perturbed problem with $\epsilon > 0$ for $t \to \infty$.

Problem 5 (extra points)
The pendulum of length L and mass m is a simple mechanical system. We want to consider it in the Cartesian coordinates. It's behaviour is described by the below DAE containing two ODEs of second order and a position constraint

$$\begin{aligned} x'' &= -\lambda x \\ y'' &= -\lambda y - g \\ 0 &= x^2 + y^2 - L^2 \end{aligned}$$

where g is the constant gravity, λ the Lagrange multiplier and is unkonwn.

a) Rewrite these two ODEs of the second order into four ODEs of the first order.

b) Compute the differentiation index of this pendulum problem.

Problem 6 (extra points)
Consider the Schmitt-trigger initial value problem

$$\begin{pmatrix} C_J & 0 & -C_J & 0 & 0 \\ 0 & C_0 & 0 & -C_0 & 0 \\ -C_J & 0 & 2C_J & -C_J & 0 \\ 0 & -C_0 & -C_J & C_0 + C_J & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} u' = - \begin{pmatrix} G_1 u_1 + (1-\alpha) g(u_1 - u_3) \\ G_2 u_2 + G_4(u_2 - u_4) + \alpha g(u_1 - u_3) \\ G_3 u_3 - g(u_1 - u_3) - g(u_4 - u_3) \\ G_4(u_4 - u_2) + (1-\alpha) g(u_4 - u_3) \\ G_5 u_5 + \alpha g(u_4 - u_3) \end{pmatrix} + \begin{pmatrix} G_1 V_{in} \\ G_2 V_{DD} \\ 0 \\ 0 \\ G_5 V_{DD} \end{pmatrix}.$$

a) Transform this problem to the semi-explicit form.

b) Find a consistent initial value for the parameters

$$G = (200, 1600, 100, 3200, 1600), \qquad C_J = 10^{-12}, \qquad C_0 = 40 \cdot 10^{-12},$$
$$g(x) = 10^{-6}\left(\exp\left(\frac{x}{0.026}\right) - 1\right), \qquad \alpha = 0.99, \qquad V_{dd} = 1$$

and the input function

$$V_{in}(t) = 2\sin(2\pi t) + 0.2\sin(20\pi t).$$

c) Solve the problem with the above parameters and initial value numerically on the interval $[0, 2]$ using an appropriate MATLAB method.

d) Solve the problem with the above parameters and initial value numerically on the interval $[0, 2]$ with the state-space method using the Runge-Kutta-4 method.

(*Mind the minus sign in the right hand side !*)