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Exercise 5 for the lecture

Numerics II

WS 2015/2016

http://numerik.mi.fu-berlin.de/wiki/WS_2015/NumericsII.php

Due: Thu, 2015-12-17

Problem 1

Let $E : \mathbb{R}^n \to \mathbb{R}$ be a continuous and coercive functional.

a) Show that E has a minimizer in $\mathbb{R}^n$.

b) Show that the minimizer is unique if E is additionally strictly convex.

c) Show that E does not necessarily have a minimizer if coercivity is dropped.

Problem 2

Let $E : \mathbb{R}^n \to \mathbb{R}$ be a convex functional and consider the associated *gradient flow*

$$x'(t) = -\nabla E(x(t)), \qquad x(0) = x_0, \tag{1}$$

where $\nabla E(x(t)) \in \mathbb{R}^n$ is the gradient of E at $x(t)$.

a) Show that $E(x(t)) \leq E(x_0)$ for all $t > 0$. Show then that even $E(x(t)) < E(x_0)$ if $\nabla E(x_0) \neq 0$.

b) Show that $f(x) = -\nabla E(x)$ is dissipative with respect to the Euclidean scalar product in $\mathbb{R}^n$.

c) Show that $x^* \in \mathbb{R}^n$ is a fixed point of (1) iff (= if and only if) x^* is a minimum of E. Furthermore, show that each isolated fixed point of (1) is stable.

d) Assume that E is strictly convex. Show that each fixed point of (1) is asymptotically stable.

Problem 3 (4 PP)
Consider the following nonlinear initial value problem

$$x'(t) = f(x), \qquad t > 0, \qquad x(0) = x_0,$$

with $x(t) \in \mathbb{R}^3$ and

$$f(x) = \begin{pmatrix} -c_1x_1 + c_3x_2x_3 \\ c_1x_1 - c_2x_2^2 - c_3x_2x_3 \\ c_2x_2^2 \end{pmatrix}.$$

a) Implement an implicit Runge-Kutta method for this equation in MATLAB as `function [x, t] = RungeKuttaNewton(f, Df, x0, I, tau, b, A, TOL)`, where `f`, `Df`, `x0`, `I`, and `tau` denote the function implementing the right hand side f, the Jacobian Df, the initial value, the time interval, the step size, and the tolerance for the nonlinear solver respectively and the Butcher scheme is given by `b, A`. The returned values `x`, `t` should contain the solution at each time step and the time steps, respectively. Use the simplified Newton method and the stopping criterion presented in the lecture to solve the nonlinear system

$$F(Z) = Z - \tau \begin{pmatrix} \sum_{j=1}^{s} a_{1j} f(x + z_j) \\ \vdots \\ \sum_{j=1}^{s} a_{sj} f(x + z_j) \end{pmatrix} = 0,$$

with $Z = [z_1, \ldots, z_s]^T \in \mathbb{R}^{3s}$.

b) Test your code for the initial value $x_0 = (1, 0, 0)^T$, the time interval $[0, 10]$, and the parameter vector $c = (1, 2, 3)$. Use the implicit Euler method, the implicit midpoint rule and the Gauß-method of order 6 for your tests. Estimate the discretization error by computing an sufficiently good reference solution numerically. Plot the estimated error as a function of the step size for all methods and for an appropiate set of step sizes. What convergence orders do you observe ?

c) Plot the total mass $x_1 + x_2 + x_3$ as function of the time. Show that the continuous flux and every Runge-Kutta scheme preserve the total mass.

Problem 4 (4 PP)

A Butcher-scheme $\begin{array}{c|c} & \mathbb{A} \\ \hline & b^T \end{array}$ with

$$A = \begin{pmatrix} a_{11} & & & \\ a_{21} & a_{22} & & \\ \vdots & \vdots & \ddots & \\ a_{s1} & a_{s2} & \ldots & a_{ss} \end{pmatrix}$$

defines a *diagonally implicit* Runge-Kutta (DIRK) method Ψ^τ of stage s. If $a_{11} = a_{22} = \ldots = a_{ss}$, Ψ^τ is called a *singly diagonally implicit* Runge-Kutta (SDIRK) method.

a) Describe the advantages of DIRK and SDIRK methods, over usual implicit Runge-Kutta methods.

b) Construct a 2-stage SDIRK method $\Psi_{2,3}^{\tau}$ of order $p = 3$.

c) Is $\Psi_{2,3}^{\tau}$ A-stable?

General remarks

You have to do the exercises in groups of up 3 people. Be prepared to demonstrate your solutions to theoretical problems at the given date in the tutorial. Solutions for programming problems have to be submitted via e-mail to `graeser@mi.fu-berlin.de` with a subject starting by [NumericsII] and denoting all members of the group. Please follow the additional advise for programming exercises on the homepage.