Fachbereich Mathematik & Informatik
Freie Universität Berlin
Prof. Dr. Carsten Gräser,

Exercise 6 for the lecture

Numerics II

WS 2015/2016

http://numerik.mi.fu-berlin.de/wiki/WS_2015/NumericsII.php

Due: Thu, 2016-01-07

Problem 1

Let $A, B \in \mathbb{R}^{n \times n}$ symmetric. Show that

$$\kappa(B^{-1}A) \leq \frac{\mu_1}{\mu_0}$$

holds, if $B \in \mathbb{R}^{n \times n}$ is a preconditioner satisfying

$$\mu_0(Bx, x) \leq (Ax, x) \leq \mu_1(Bx, x) \quad \forall x \in \mathbb{R}^n$$

for some $0 < \mu_0, \mu_1 \in \mathbb{R}$.

Problem 2

Show that, if A is strongly diagonal dominant, i.e.,

$$\sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| < |a_{ii}| \quad \forall i = 1, \dots n,$$

the Jacobi method is globally convergent.

Problem 3

The symmetric Gauß-Seidel method for the solution of a linear system with a s.p.d. matrix is obtained by applying one normal Gauß-Seidel step and one Gauß-Seidel step with the components in reversed order alternatingly. Give the iteration matrix and the generated preconditioner of the symmetric Gauß-Seidel method and show that the preconditioner is s.p.d..

Please turn over.

Problem 4 (8 PP)
Consider the the linear system

$$AU = b \tag{1}$$

with the symmetric positive definite matrix $A \in \mathbb{R}^{n,n}$ and $b \in \mathbb{R}^n$.

a) Compute an upper bound for the convergence rate of the Jacobi method applied to the linear system (1) with the matrix A obtained by a finite difference discretization of the Poisson equation using a uniform grid on $[0,1] \times [0,1]$ given in the lecture.

b) Implement the Jacobi and the Gauß-Seidel methods in `matlab` as

```
function [u, uk] = Jacobi(A, b, u0, tol, uexact)
```

and

```
function [u, uk] = GaussSeidel(A, b, u0, tol, uexact).
```

`u`, `uk`, `A`, `b`, `u0`, `tol`, and `uexact` denote the last iterate, a vector containing all iterates, the system matrix, the right hand side, the initial iterate, the error tolerance, and the exact solution, respectively. The iteration should stop if the energy norm $\|\cdot\|_A = \langle A\cdot, \cdot\rangle^{0.5}$ of the error is smaller than the tolerance. Test your programm with the matrix of part a) and the right hand side $b = AU$ where U is the point wise evaluation of $(x_1 - x_1^2)(x_2 - x_2^2)$ for `u0` $= 0$, `tol` $= 10^{-8}$ and various choices of n. Plot the error over the number of iteration steps and compute the average convergence rate for each choice of n.

General remarks

You have to do the exercises in groups of up 3 people. Be prepared to demonstrate your solutions to theoretical problems at the given date in the tutorial. Solutions for programming problems have to be submitted via e-mail to `graeser@mi.fu-berlin.de` with a subject starting by [NumericsII] and denoting all members of the group. Please follow the additional advise for programming exercises on the homepage.