Fachbereich Mathematik & Informatik
Freie Universität Berlin
Prof. Dr. Carsten Gräser,

Exercise 9 for the lecture

Numerics II

WS 2015/2016

http://numerik.mi.fu-berlin.de/wiki/WS_2015/NumericsII.php

Due: Thu, 2016-02-04

Problem 1

Let $A \in \mathbb{R}^{n\times n}$ be symmetric and positive definite, $b \in \mathbb{R}^n$, and $J(x) = \frac{1}{2}a(x,x) - l(x)$ for the bilinear form $a(x,y) = \langle Ax, y\rangle$ and the linear functional $l(x) = \langle b, x\rangle$. We consider the minimization problem

$$x^* \in \mathbb{R}^n : \qquad J(x^*) \leq J(x) \qquad \forall x \in \mathbb{R}^n.$$

Let $T = (b^1, \dots, b^m) \in \mathbb{R}^{n,m}$ be any matrix with columns $b^1, \dots, b^m \in \mathbb{R}^n$ and $V = \operatorname{span}\{b^1, \dots, b^m\} = \operatorname{ran} A$.

a) Show that $A_V = T^T A T \in \mathbb{R}^{m\times m}$ is symmetric positive semi-definite.

b) Show that $A_V = T^T A T \in \mathbb{R}^{m\times m}$ is symmetric positive definite if and only if $\{b^1, \dots, b^m\}$ are linearly independent.

c) Show that for any $w \in \mathbb{R}^n$ the minimization problem

$$v \in V : \qquad J(w+v) \leq J(w+x) \qquad \forall x \in V \tag{1}$$

is equivalent to the variational equation

$$v \in V : \qquad a(v,x) = l(x) - a(w,x) \qquad \forall x \in V$$

and $v = T\tilde{v}$ for the linear system

$$A_V \tilde{v} = T^T(b - Aw). \tag{2}$$

d) Show that for any $w \in \mathbb{R}^n$ there exists a unique solution v of (1) and a solutions $\tilde{v}_0$ of (2). Furthermore show that the set of all solutions of (2) is given by the affine subspace $\tilde{v}_0 + \ker T$.

Problem 2

Let $A \in \mathbb{R}^{n\times n}$ be symmetric and positive definite, $b \in \mathbb{R}^n$, and $J(x) = \frac{1}{2}a(x,x) - l(x)$ for the bilinear form $a(x,y) = \langle Ax, y\rangle$ and the linear functional $l(x) = \langle b, x\rangle$. We consider the minimization problem

$$x^* \in \mathbb{R}^n : \qquad J(x^*) \leq J(x) \qquad \forall x \in \mathbb{R}^n.$$

For $k = 1, \dots, j$ consider subspaces $V_k = \operatorname{span}\{d_k^1, \dots d_k^{m_k}\} \subset \mathbb{R}^n$. For $m = \sum_{k=1}^j m_k$ let

$$T = (T_1, \dots, T_j) \in \mathbb{R}^{n\times m}, \qquad\qquad T_k = (d_k^1, \dots d_k^{m_k}) \in \mathbb{R}^{n\times m_k}$$

and consider a block left-diagonal-right splitting $\tilde{A} = \tilde{L} + \tilde{D} + \tilde{R}$ of $\tilde{A} = T^T A T$ where $\tilde{B}$ is the block diagonal matrix with j diagonal blocks of size $(m_1 \times m_1), \dots, (m_j \times m_j)$.

a) Show that one step of the parallel subspace correction method

   1. Given: u^ν
   2. For $k = 1, \dots, j$ do
      2.1. Compute $v^k = \operatorname*{arg\,min}_{v\in V_k} J(u^\nu + v)$
   3. Set $u^{\nu+1} = u^\nu + \sum_{k=1}^j v^k$

   is equivalent to a block-Jacobi step in coordinates wrt the columns of T, i.e.,

   1. Given: u^ν
   2. Solve $\tilde{D}\tilde{v} = T^T(b - Au^\nu)$
   3. Set $u^{\nu+1} = u^\nu + T\tilde{v}$

b) Show that one step of the sequential subspace correction method

   1. Given: u^ν
   2. Let $w^0 = u^\nu$
   3. For $k = 1, \dots, j$ do
      3.1. Compute $v^k = \operatorname*{arg\,min}_{v\in V_k} J(w^{k-1} + v)$
      3.2. Set $w^k = w^{k-1} + v^k$
   4. Set $u^{\nu+1} = w^j$

   is equivalent to a block-Gauß–Seidel step in coordinates wrt the columns of T, i.e.,

   1. Given: u^ν
   2. Solve $(\tilde{D} + \tilde{L})\tilde{v} = T^T(b - Au^\nu)$
   3. Set $u^{\nu+1} = u^\nu + T\tilde{v}$