Institut für Mathematik
Freie Universität Berlin
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Numerics II
WS 2015/2016

Name: ______________________________ Matr.-Nr.: ____________________

Course of studies: ◯ Mathematics ◯ Bioinformatics ◯ BMS ◯ Computer Science ◯ other:

Intended degree: ◯ Diplom ◯ Lehramt(Staatsexamen) ◯ other: ◯ Bachelor(Mono) ◯ Bachelor(Kombi, Lehramt) ◯ Master

Note your name on all sheets you hand in and staple them together. Please do not use a pencil. You are allowed to use all your written documents, books, and a non-programmable calculator. Other electronic devices are not allowed. The exam consists of 3 pages on two sheets.

If you want to find your results on the lecture web page next to your matriculation number sign the following declaration:

I agree with the publication of my results next to my matriculation number on the lecture web page.

______________________________ (sign here)

Problem	I	II.1	II.2	II.3	$\sum$
Points					

Good luck!

Part I (8 points)

For each of the following statements check if it is 'true' or 'false', note your answer, and explain it by one sentence or a counterexample.

You get one point for each statement where your answer and explanation is correct. If either an answer is not correct or no correct explanation is given, you get zero points for the corresponding statement.

a) The initial value problem

$$x'(t) = \sin^2(x(t)) \qquad \forall t \in (0, T], \qquad x(0) = x_0$$

has a unique solution for all $T > 0$ and all initial values $x_0 \in \mathbb{R}$.

b) The discrete flow operator Ψ^τ of a Gauß-method is non-expansive for all autonomous ODEs with dissipative right hand side and $\tau > 0$.

c) Gauß-methods are A-stable.

d) An A-stable Runge–Kutta method can be applied to any linear, autonomous ODE $x' = Ax$ without any time step restriction.

e) The linear iterative method $x^{k+1} = x^k + B^{-1}(b - Ax^k)$ for symmetric positive definite matrices $A, B \in \mathbb{R}^{n\times n}$ converges to the solution x^* of $Ax^* = b$ if

$$\langle Ax, x\rangle \leq \langle Bx, x\rangle \qquad \forall x \in \mathbb{R}^n.$$

f) Consider a linear system $Ax = b$ with symmetric, positive definite $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$, and $J(y) = \frac{1}{2}\langle Ay, y\rangle - \langle b, y\rangle$. For any $x^0 \in \mathbb{R}^n$, the iterates $x^0, x^1, x^2, \ldots$ provided by a parallel directional correction method, applied to this system, satisfy $J(x^0) \geq J(x^1) \geq J(x^2) \geq \cdots \geq J(x)$.

g) If the GMRes method is applied to a linear system $Ax^* = b$ with a symmetric positive definite matrix A, then it is equivalent to the CG method applied to this system.

h) The phase flow $\Phi^t : \mathbb{R}^2 \to \mathbb{R}^2$ of a Hamiltonian system has the property $\det(D\Phi^t) = 0$.

Please turn over

Part II (16 points)

Complete **all** of the following exercises!

Problem 1 (2+2+2+2 points)

a) Check whether $x^* = 0$ is a stable or asymptotically stable fixed point of the following ODE and justify your results:

$$x' = \begin{pmatrix} -x_1 + x_2 + x_1 x_2 \\ x_1^2 x_2^2 - x_2 \end{pmatrix}.$$

b) Check whether $x^* = 0$ is a stable or asymptotically stable fixed point of the following ODE and justify your results:

$$x' = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} x. \tag{1}$$

c) Compute the stability function of the Heun-method, i.e., the Runge–Kutta method given by the Butcher scheme

$$\begin{array}{c|cc} & 0 & 0 \\ & \frac{2}{3} & 0 \\ \hline & \frac{1}{4} & \frac{3}{4} \end{array}.$$

d) Which time step restriction guarantees that the Heun-method inherits the stability properties of the ODE (1)?

Problem 2 (1+2+1 points)

For the linear system $Ax = b$ with symmetric and positive definite matrix $A \in \mathbb{R}^{n \times n}$ and $b \in \mathbb{R}^n$ let $A = D + L + R$ be the splitting of A into diagonal, left, and right part and $B = (D + L)D^{-1}(D + R)$.

a) Show that $B = A + LD^{-1}R$.

b) Show that $\lambda_{\max}(B^{-1}A) \le 1$. (Hint: Use a))

c) Show that the iterative method $x^{k+1} = x^k + B^{-1}(b - Ax^k)$ converges. (Hint: Use b))

Problem 3 (2+2 points)

Consider the conservation law

$$x'' = -x \tag{2}$$

with $x \in C^2([0, \infty), \mathbb{R}^d)$.

a) Rewrite (2) as a Hamiltonian system with Hamiltonian $H(p, q)$.

b) Show that $H(p, q)$ is conserved throughout the evolution, i.e., $H(x'(t), x(t))$ is constant in t for any solution x of (2).

End of the exam