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Exercise sheet #1

Numerics of stochastic differential equations

Wintersemester 2015/16

'The different branches of mathematics:
ambition, distraction, uglification, and derision.'
Lewis Carroll

Hand in until: Monday, 10th November, 10:15am

Exercise 1. (Gaussians I)

Let X_1, X_2 be two Gaussian random variables that have a joint distribution $N(\mu, \Sigma)$, with mean $\mu \in \mathbb{R}^2$ and covariance matrix $\Sigma \in \mathbb{R}^{2\times 2}$ where we assume that Σ is positive definite. The density of the joint distribution with respect to the Lebesgue measure on $\mathbb{R}^2$ is given by

$$\rho(x) = (\det(2\pi\Sigma)^{-1/2} \exp\left(-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\right), \quad x = (x_1, x_2) \in \mathbb{R}^2$$

Show that the conditional probability measure $\mu_1^a(A) := P(X_1 \in A|X_2 = a)$, $A \in \mathcal{B}(\mathbb{R})$ has a Lebesgue density $\rho_1^a(x_1)$ for every $a \in \mathbb{R}$, and compute $\rho_1^a(x_1)$.

Exercise 2. (Gaussians II)

Consider the time–discrete stochastic process $(X_n)_{n\geq 0}$ defined by

$$X_{n+1} = X_n + \sigma\xi_{n+1}, \quad X_0 = 0,$$

with $\sigma \in \mathbb{R}$ and $(\xi_n)_{n\in\mathbb{N}}$ i.i.d. normalized Gaussian random variables (i.e. $\mathbb{E}(\xi_n) = 0$ and $\mathrm{Var}(\xi_n) = 1$). Specify the distribution of $(X_0 \ldots, X_k)$ for any given $k \in \mathbb{N}$. (*Hint: cf. Exercise 1.*)

Exercise 3. (Brownian motion)

Write a program (e.g. in Matlab) that generates $M = 1\,000$ realizations of the stochastic process

$$X_{n+1}^{\Delta t} = X_n^{\Delta t} + \sqrt{\Delta t}\,\xi_{n+1}, \quad X_0^{\Delta t} = 0\,.$$

where the ξ_n are either i.i.d. $N(0,1)$ random variables or i.i.d. with $P(\xi_n = \pm 1) = 1/2$.

(a) For $\Delta t = 1/N$ with $N \in \{100, 1\,000, 10\,000\}$, plot a histogram of the distribution of $X_N^{\Delta t}$ for both choices of the ξ_n and describe your observation.

(b) Define the stopping time $\tau = \inf\{n\colon X_n^{\Delta t} \geq 1\}$ to be the first hitting time of the set $\{x \geq 1\}$ and estimate the mean first hitting time $\mathbb{E}(\tau)$ for $N = 10\,.000$, using your program.

(c) Now consider the process

$$X_{n+1}^{\Delta t} = X_n^{\Delta t} + (\Delta t)^\alpha \xi_{n+1}, \quad X_0^{\Delta t} = 0,$$

with $\alpha = 0.4$ and $\alpha = 0.6$, and Δt as in (a). Compare the behaviour of the distribution of $X_N^{\Delta t}$ to the one observed in (a) and give an interpretation of the result.

Exercise 4. (Quadratic variation)
Given a stochastic process $(X_s)_{s\geq 0}$ on a probability space $(\Omega, \mathcal{F}, P)$ assuming values in $\mathbb{R}$, the *p-th variation process* is defined by

$$[X]_t^{(p)}(\omega) = \lim_{\Delta\to 0} \sum_{t_k\leq t} |X_{t_{k+1}}(\omega) - X_{t_k}(\omega)|^p\,, \quad p > 0\,.$$

where $0 = t_1 < t_2 < \ldots < t_n = t$ is any partition of the interval $[0, t]$ with $\Delta t_k = t_{k+1} - t_k$ and $\Delta = \sup\{\Delta t_k : k \geq 0\}$. The limit is understood "in probability". If $p = 1$ this process is called the *total variation process*, for $p = 2$ it is called the *quadratic variation process*.

(a) Show that the quadratic variation of the one-dimensional Wiener process W_t is given by

$$[W]_t^{(2)} = t\,.$$

(*Hint: Define* $\Delta W_k = W_{t_{k+1}} - W_{t_k}$ *and set* $Y_t(\omega) = \sum_{t_k\leq t}(\Delta W_k(\omega))^2$. *Then show that*

$$\mathbb{E}\Big[\Big(\sum_{t_k\leq t}(\Delta W_k)^2 - t\Big)^2\Big] = 2\sum_{t_k\leq t}(\Delta t_k)^2$$

and deduce that $Y_t \to t$ *in* $L^2(\Omega, P)$ *as* $\Delta \to 0$.)

(b) Prove that almost all paths of Brownian motion do not have a bounded total variation on $[0, t]$ (*Hint: Show that a continuous real function with bounded total variation has zero quadratic variation.*)