Fachbereich Mathematik & Informatik
Freie Universität Berlin
Carsten Hartmann, Lara Neureither

Exercise sheet #2

Numerics of stochastic differential equations

Wintersemester 2015/16

'Tout le monde y croit cependant [que les erreurs sont normalement distribuées], me disait un jour M. Lippmann, car les expérimentateurs s'imaginent que c'est un théorème de mathématiques, et les mathématiciens que c'est un fait expérimental.'
Henri Poincaré

Hand in until: Tuesday, 1st December, 10:15am

Exercise 1. (Integration à la Itô)
Unless otherwise stated, $(W_t)_{t\geq 0}$ always denotes a one-dimensional standard Brownian motion on $(\Omega, \mathcal{E}, P)$, and $f, g \in \mathbb{L}^2([0,T])$ are Itô-integrable processes for all $T > 0$.

(a) Prove directly from the definition of the Itô integral that

$$\int_0^t s dW_s = tW_t - \int_0^t W_s ds\,, \quad t \in (0,T]\,.$$

Discuss the relation with the Payley-Wiener-Zygmund (PWZ) integral, in particular, discuss possible extensions of the PWZ definition so as to include the above relation.

(b) Use the Itô formula to calculate

$$I_T = \int_0^T W_s dW_s\,.$$

(c) Let $\{\mathcal{F}_t : t \geq 0\}$ be the filtration generated by W_t. Show that

$$M_t(\omega) = \int_0^t f(s,\omega) dW_s(\omega)$$

is $\mathcal{F}_t$-measurable for any $t \geq 0$, and use this result to prove that M_t is a martingale with respect to the filtration generated by W_t.

Exercise 2. (Riemann sums)
Let $\Delta = \{0 = t_0 < t_1 < \ldots < t_N = T\}$ be a partition of the interval $[0,T]$ with mesh size $|\Delta| = \sup\{t_{k+1} - t_k : 0 \leq k \leq N-1\}$. Further define the collection $\{\tau_k\}$ of intermediate points $\tau_k = (1-\lambda)t_k + \lambda t_{k+1}$, $k = 0, 1, 2, \ldots, N-1$ for any $\lambda \in [0,1]$ and consider the Riemann sum approximation

$$R_{\Delta,\lambda} = \sum_{k=0}^{N-1} W_{\tau_k} \left(W_{t_{k+1}} - W_{t_k} \right)$$

of I_T from Exercise 1. Show that, as $|\Delta| \to 0$,

$$R_{\Delta,\lambda} \to \frac{1}{2} W_T^2 + \left(\lambda - \frac{1}{2} \right) T \quad \text{in } L^2(\Omega, P)\,.$$

For which choice of λ does $R_{\Delta,\lambda}$ converge to the Itô integral?

Exercise 3. (Ornstein-Uhlenbeck process I)
Let W_t be a standard Brownian motion in $\mathbb{R}^n$, and consider the SDE

$$dX_t = AX_t dt + BdW_t, \quad X_0 = x$$

where A and B are $(n \times n)$-matrices and $x \in \mathbb{R}^n$.

(a) Show that the solution of the SDE is given by the variation-of-constants formula

$$X_t = e^{At}x + \int_0^t e^{A(t-s)} \overset{B}{\wedge} dW_s\,.$$

(Hint: Use the n-dimensional Itô formula with $f(x,t) = e^{-At}x$.)

(b) Use (a) to compute $\mu_t = \mathbb{E}[X_t]$ and the covariance matrix Σ with components

$$\Sigma_{ij} = \mathbb{E}[(X_t^i - \mu_t^i)(X_t^j - \mu_t^j)]$$

where X_t^i and μ_t^i are the i-th components of X_t and μ_t respectively.

(c) Suppose that all eigenvalues of A have strictly negative real part and define Σ_∞ to be the limit of $\mathbb{E}[(X_t - \mu_t)(X_t - \mu_t)^T]$ as $t \to \infty$. Show that $A\Sigma_\infty + \Sigma_\infty A^T = -BB^T$.

(Hint: Use partial integration.)

Exercise 4. (Ornstein-Uhlenbeck-process II)
Write a program to numerically integrate the Ornstein-Uhlenbeck-process X_t from Exercise 2 for $n = 2$ using the Euler-Maruyama scheme, with $B = I$ the (2×2) identity matrix and A being any of the following matrices:

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

(a) Plot typical realizations of the numerically integrated process X_t with time step $\Delta t = 0.01$ for all $A = A_i$ over a time interval $[0, 10]$ compare it with the 'deterministic' process $(B = 0)$.

(b) Explain the qualitative behaviour observed for the different choices of $A = A_i$.

(Hint: eigenvalues.)