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Exercise sheet #3

Numerics of stochastic differential equations

Wintersemester 2015/16

'One of the chief duties of a mathematician in acting as an advisor to scientists is to discourage them from expecting too much of mathematicians.'
Norbert Wiener

Hand in until: Tuesday, 15th December, 10:15am

Exercise 1. (Brownian motions)

(a) Let $(W_t)_{t\geq 0}$ be one-dimensional Brownian motion. For $a, b > 0$, consider a two-dimensional stochastic process $(Z_t)_{t\geq 0}$, $Z_t = (X_t, Y_t))$ on the ellipse $\{(x,y)\colon (x/a)^2 + (y/b)^2 = 1\}$ defined by

$$Z_t = (a\cos W_t,\, b\sin W_t)\,.$$

Show that Z_t is the unique strong solution of the SDE

$$dX_t = -\frac{1}{2}X_t dt - \frac{a}{b}Y_t dW_t\,, \quad X_0 = a\,,$$
$$dY_t = -\frac{1}{2}Y_t dt + \frac{a}{b}X_t dW_t\,, \quad Y_0 = 0\,.$$

(b) For $a, b \in \mathbb{R}$, let $(B_t)_{0\leq t\leq 1}$ be the solution of the SDE

$$dB_t = \frac{b - B_t}{1-t}dt + dW_t\,, \quad B_0 = a\,.$$

Show that

$$B_t = a(1-t) + bt + (1-t)\int_0^t \frac{dW_s}{1-s}\,, \quad 0 \leq t \leq 1\,,$$

and prove that $B_t \to b$ in probability as $t \uparrow 1$.

Exercise 2. (Heun's method)

Consider the process $(X_t)_{t\geq 0}$ in $\mathbb{R}$ given by the SDE

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t\,, \quad X_0 = x\,,$$

with b, σ satisfying the usual Lipschitz and growth conditions, $x \in \mathbb{R}$, and W_t denoting Brownian motion in $\mathbb{R}$. Let $(\tilde{X}_n)_{n\in\mathbb{N}_0}$ be the sequence of iterates of the *stochastic Heun method*

$$Y_{n+1} = \tilde{X}_n + \Delta t b(\tilde{X}_n) + \sqrt{\Delta t}\sigma(\tilde{X}_n)\eta_{n+1}\,, \quad \tilde{X}_0 = x\,,$$
$$\tilde{X}_{n+1} = \tilde{X}_n + \frac{\Delta t}{2}\Big(b(\tilde{X}_n) + b(Y_{n+1})\Big) + \frac{\sqrt{\Delta t}}{2}\Big(\sigma(\tilde{X}_n) + \sigma(Y_{n+1})\Big)\eta_{n+1}\,,$$

for a time step $\Delta t > 0$ and $(\eta_n)_{n\in\mathbb{N}}$ being a sequence of i.i.d. normalized Gaussian random variables.

Show that Heun's method is neither strongly or weakly convergent and explain your findings.

(*Hint:* Consider the example $dX_t = 2X_t dW_t$ with $X_0 = 1$.)

Exercise 3. (Simple Euler scheme)

Consider the SDE from Exercise 2. Let $T > 0$. The *simple stochastic Euler scheme* approximates $(X_t)_{0\le t\le T}$ by

$$\tilde{X}_{n+1} = \tilde{X}_n + \Delta t b(\tilde{X}_n) + \sqrt{\Delta t}\sigma(\tilde{X}_n)\xi_{n+1}\,, \quad \tilde{X}_0 = x\,,\ n \in \mathbb{N}_0$$

where $\Delta t_{>}0$ is the step size and $(\xi_n)_{n\in\mathbb{N}_0}$ is an i.i.d. sequence of random variables on a probability space $(\Omega, \mathcal{E}, P)$ with $P(\xi_n = \pm 1) = 1/2$.

Prove that the simple stochastic Euler method satisfies Itô's formula: For $g \in C^2(\mathbb{R})$, we have

$$dg(\tilde{X}_n) = g'(\tilde{X}_n)(\tilde{X}_n - \tilde{X}_{n-1}) + \frac{(\sigma(\tilde{X}_n))^2}{2} g''(\tilde{X}_n)\Delta t + o(\Delta t)$$

where $f = o(\Delta t)$ means that $f/\Delta t \to 0$ as $\Delta t \to 0$ where the limit holds a.s.

Exercise 4. (Simple Euler scheme, continued)

Use the simple Euler scheme to numerically calculate a realization of the OU process

$$dX_t = -X_t dt + dW_t\,, \quad X_0 = 1\,.$$

(a) Fix the step size $\Delta t = 0.01$ and $n = 0, 1, \ldots, N$ with $N = 100\,000$ and plot a histogram of $\tilde{X}_N$ and compare it to the exact distribution of X_T for $T = N\Delta t$. Describe and interpret your observations? (*Hint:* Pick an appropriate measure of discrepancy to compare the two probability distributions.)

(b) The simple stochastic Euler is weakly convergent of order 1, i.e.

$$\max_{0\le k\le N} |\mathbb{E}(f(X_{k\Delta t})) - \mathbb{E}(f(\tilde{X}_k))| \le C\Delta t$$

for some $C \in (0, \infty)$ independent of Δt and $f \in C_b(\mathbb{R})$. (It does not converge strongly though.) Describe a Monte Carlo methods to calculate an approximation of

$$\varphi(x) = \mathbb{E}(f(X_T)|X_0 = x)\,, \quad .$$

for $T > 0$ and any given continuous and bounded function $f\colon \mathbb{R} \to \mathbb{R}$. Numerically compute the order of convergence for $f = \chi_{[-2,2]^c}$ and explain your observations. .