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Exercise sheet #5

Numerics of stochastic differential equations

Wintersemester 2015/16

'It is the story that matters, not just the ending.'
Paul Lockhart, A Mathematician's Lament (2002)

Hand in until: Tuesday, 26th January, 10:15am

Exercise 1. (Continuous semigroups III)
Let $(K, \|\cdot\|)$ be a Banach space with norm $\|\cdot\|$. A *strongly continuous contraction semigroup* is a one-parameter family of operators $S_t\colon K \to K$ such that

(i) $S_0 = \mathrm{Id}$ and $S_{t+u} = S_t \circ S_u$ for all $t, u > 0$,

(ii) $\|S_t f - f\| \to 0$ as $t \to 0$ (continuity),

(iii) $\|S_t\| \le 1$ for all $t > 0$ (contractivity).

Consider a real-valued, time-homogeneous diffusion process $(X_t)_{t\ge 0}$ on a probability space $(\Omega, \mathcal{E}, P)$ with smooth transition density $p(t, x, \cdot)$, such that for all $t \ge 0$

$$P(X_t \in B | X_0 = x) = \int_B p(t, x, y)dy\,, \quad B \in \mathcal{B}(\mathbb{R}^n)\,.$$

Show that $S_t f = \mathbb{E}_x[f(X_t)]$ defines a strongly continuous semigroup on $(C_0(\mathbb{R}^n), \|\cdot\|_\infty)$ where $C_0(\mathbb{R}^n)$ is the space of continuous functions with compact support and $\|\cdot\|_\infty$ is the supremum norm.

Exercise 2. (Fokker-Planck equation)
Let $(X_t)_{t\ge 0}$ be a diffusion process on $\mathbb{R}^n$ with generator

$$L = \frac{1}{2}a(x)\colon \nabla^2 + b(x)\cdot\nabla\,,$$

where $a(x) = \sigma(x)\sigma(x)^T > 0$ has uniformly bounded inverse for all $x \in \mathbb{R}^n$ and b, σ satisfy the usual Lipschitz and growth conditions. Further let $\rho(x, t)$ be the classical solution of the Fokker-Planck equation

$$\frac{\partial\rho}{\partial t} = \frac{1}{2}\nabla^2\colon(a\rho) - \nabla\cdot(b\rho)\,, \quad \rho(x, 0) = \rho_0(x)\,.$$

(a) Show that

$$\|\rho_0\|_{L^1(\mathbb{R}^n)} = 1 \qquad \Longrightarrow \qquad \|\rho(\cdot, t)\|_{L^1(\mathbb{R}^n)} = 1 \quad \forall t > 0\,.$$

(b) Now suppose that X_t is confined to an open and bounded subset $O \in \mathbb{R}^n$ with smooth (e.g. C^∞) boundary ∂O. Modify the above Fokker-Planck equation, such that

$$\|\rho_0\|_{L^1(O)} = 1 \qquad \Longrightarrow \qquad \|\rho(\cdot, t)\|_{L^1(O)} = 1 \quad \forall t > 0\,.$$

i.e., such that the total probability is conserved, and justify your choice. (*Hint: boundary conditions*)

Exercise 3. (Invariant distribution)

Let X_t be the one-dimensional Ornstein-Uhlenbeck process

$$dX_t = -\mu X_t dt + \sigma dW_t\,, \quad X_0 = x$$

with $\mu, \sigma > 0$.

(a) Compute the stationary distribution of X_t and show that it is unique.

(b) Let $\hat{X}_n$ be the Euler-Maruyama discretisation for a time step $\Delta t \in (0, 2/\mu)$. Compute the distribution of $\hat{X}_n$ and its limit as $n \to \infty$. Compare with (a).

Exercise 4. (Heat equation)

Consider the scalar process $X_t = X_0 + W_t$ with initial conditions X_0 that are uniformly distributed on the interval $[-1, 1] \subset \mathbb{R}$. Let $\rho(\cdot, t)$ be the law of X_t that is governed by the heat equation

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right) \rho(x, t) = 0\,, \quad \rho(\cdot, 0) = \mathcal{U}([-1, 1])\,.$$

Let $T > 0$. We want to numerically compute the probability $p_{B,T} = P(X_T \in B)$ for some measurable set $B \in \mathcal{B}(\mathbb{R})$ and under the assumption $X_0 \sim \rho(\cdot, 0)$.

(a) Let $T = 1$ and $B = [2, 3]$. Estimate $p_{[2,3],1}$ by Monte Carlo, based on sufficiently many realisations of the process X_T. Explain your choice of the simulation parameters.

(b) Calculate $p_{[2,3],1}$ by solving the heat equation. Proceed as follows: As a first step, discretise the heat equation in space using an equidistant grid on $[-5, 5]$ with suitable boundary conditions, then solve the resulting ODE using a suitable time stepping scheme. Compare with (a).