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2nd Exercise for the Lecture

Numerics IV

WiSe 2015/2016

http://numerik.mi.fu-berlin.de/wiki/WS_2015/NumerikIV.php

Discussion on: Tue, November 10th, 2015

Exercise 1 (Orientability of Curves)
Prove or disprove that the curve $\Gamma = (X(\theta), Y(\theta))_{\theta\in[0,1]} \subseteq \mathbb{R}^2$ is orientable for all functions $X, Y \in C^1([0,1])$ if $X'(\theta)^2 + Y'(\theta)^2 \neq 0$ holds for all $\theta \in [0,1]$.

Remark: A curve $\Gamma \subseteq \mathbb{R}^2$ is called orientable if there exists a continuous normal vector field $\nu\colon \Gamma \to \mathbb{R}^2$.

Exercise 2 (Curvature of Curves)
Let $X, Y \in C^2([0,1])$ and $\Gamma = (X(\theta), Y(\theta))_{\theta\in[0,1]}$. Given $X'(\theta)^2 + Y'(\theta)^2 \neq 0$ for all $\theta \in [0,1]$, show that the curvature κ of Γ is given by

$$\begin{aligned} \kappa\colon [0,1] &\longrightarrow \mathbb{R} \\ \theta &\longmapsto \frac{X'(\theta)Y''(\theta) - Y'(\theta)X''(\theta)}{((X'(\theta)^2 + Y'(\theta)^2)^{\frac{3}{2}}} \end{aligned}$$

where $\kappa(\theta)$ denotes the curvature of Γ in the point $(X(\theta), Y(\theta)) \in \Gamma$.

Exercise 3 (Mean Curvature Flow of Unit Circle)
Let $X_0 = (\cos(\theta), \sin(\theta))_{\theta\in[0,2\pi]}$. Show that the mean curvature flow of X_0 on $[0, \frac{1}{2}]$ is given by

$$\begin{aligned} X\colon \left[0, \tfrac{1}{2}\right] \times [0,1] &\longrightarrow \mathbb{R}^2 \\ (t, \theta) &\longmapsto \sqrt{1-2t}\begin{pmatrix}\cos(\theta)\\ \sin(\theta)\end{pmatrix}. \end{aligned}$$

Please turn over...

Exercise 4 (Stationary Solutions of the Mean Curvature Flow are Local Minimizers of the Area Functional (1D))
In this exercise, denote $\Gamma(f) := (x, f(x))_{x\in[0,1]}$ for all $f \in H^1([0,1])$. Assume there is a stationary solution of the mean curvature flow that can be represented as $\Gamma(f^*)$ for some $f^* \in H^2([0,1]) \cap H_0^1([0,1])$.
Show that f^* is a minimizer of

$$\min_{f\in H_0^1([0,1])} |\Gamma(f)|$$

where $|\Gamma(f)|$ denotes the arc length of $\Gamma(f)$. Conclude that f^* must be constant.

Remark:

- We call a solution $(X(t))_{t\in[0,T]}$ of a flow *stationary* if $X_t(t) = 0$ holds for all $t \in [0,T]$. In particular, we can then identify the solution with $X(0)$ because $(X(t))_{t\in[0,T]}$ is constant.
- This result also holds for general curves Γ that do not necessarily admit a graph representation.

Have fun!